Enhanced Scenario Analysis

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Enhanced Scenario Analysis

- Mean-variance analysis versus scenario analysis
- Scenario analysis process
- Statistical probabilities of economic scenarios
- Case study
- Scenario modification
- Concluding comments

Mean-Variance Analysis versus Scenario Analysis

Mean-variance analysis

- Requires estimates of expected returns, standard deviations, and correlations
- Yields efficient portfolios along with estimates of their expected returns and standard deviations
- Advantages:
  - Implicitly accounts for all potential scenarios along with their probabilities of occurrence
  - Statistically rigorous
- Disadvantages:
  - Inputs and outputs are unintuitive
  - Required inputs are difficult to estimate

Mean-Variance Analysis versus Scenario Analysis

Scenario analysis

- Requires specification of prospective scenarios along with their probabilities of occurrence
- Yields target portfolio
- Advantages:
  - Intuitive
  - Straightforward to implement
- Disadvantages:
  - Accounts for only a small subset of potential scenarios
  - Reliant on subjective views
  - Lacks statistical rigor

Scenario Analysis Process

1. Specify scenarios as projections of economic variables
2. Assign probabilities to prospective scenarios
3. Map economic variables onto asset class expected returns
4. Compute probability-weighted expected returns for each asset class
5. Specify alternative portfolio choices
6. Compute each portfolio's expected return based on probability-weighted asset class expected returns
7. Select best portfolio

Statistical Probabilities of Prospective Scenarios

The Mahalanobis distance

- The Mahalanobis distance was introduced by an Indian statistician in 1927 and enhanced in 1936 to analyze distances and resemblances of skulls among various castes in India.
- Mahalanobis observed that a half inch difference in nasal length was more significant than a half inch difference in skull length.
- He developed an equation to standardize differences in various skull characteristics based on their standard deviations and correlations.
- This formula was rediscovered in 1999 to measure the statistical unusualness of a cross section of asset class returns as an indication of financial turbulence.
- In our research, we adapt the Mahalanobis distance to measure the statistical unusualness of a prospective set of economic variables, which we then convert into probabilities of occurrence.

Chow, G., E. Jacquier, K. Lowry, and M. Kritzman. 1999. "Optimal Portfolios in Good Times and Bad." *Financial Analysts Journal*, vol. 55, no. 3 (May/June):65–73.

Mahalanobis, P.C. 1927. "Analysis of Race-Mixture in Bengal." *Journal of the Asiatic Society of Bengal*, vol. 23:301–333.

Mahalanobis, P. C. 1936. "On the Generalised Distance in Statistics." *Proceedings of the National Institute of Sciences of India*, vol. 2, no. 1:49–55.

Statistical Probabilities of Prospective Scenarios

Left half and right half are skulls from two different castes.

Mahalanobis (1927) used seven skull characteristics to analyze race-mixture in Bengal:

- Head length
- Head breadth
- Nasal length
- Nasal breadth
- Cephalic index
- Nasal index
- Stature

Source: Mahalanobis, P.C. 1927. “Analysis of Race-Mixture in Bengal.” *Journal of the Asiatic Society of Bengal*, vol. 23:301–333.

Statistical Probabilities of Prospective Scenarios

The Mahalanobis distance

$$d = (x - \gamma)'\Sigma^{-1}(x - \gamma)$$

d = the Mahalanobis distance

x = the values of a set of variables used to characterize a prospective scenario

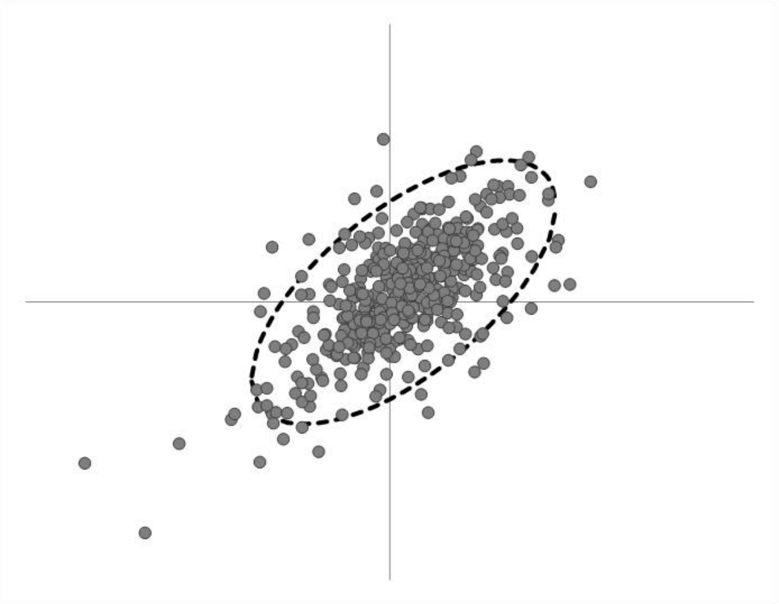
γ = the prevailing values of the economic variables

Σ = the historical covariance matrix of changes in values for those variables.

- The term $(x - \gamma)$ captures how unusual each projected economic variable is in isolation given its prevailing value.
- By multiplying $(x - \gamma)$ by the inverse of the covariance matrix, we capture the unusualness of the co-occurrence of the projected variable changes.
- This multiplication also renders the measure scale independent.

Statistical Probabilities of Prospective Scenarios

Scatter Plot of Two Hypothetical Economic Variables



- Each dot represents the joint values of the two economic variables for a given observation.
- The center of the ellipse represents the prevailing values of the two variables.
- The observations within the ellipse represent reasonably plausible combinations, because the observations are not particularly distant from the prevailing values.
- The observations outside the ellipse are statistically unusual and therefore likely to characterize much less plausible combinations.

Statistical Probabilities of Prospective Scenarios

Scale Independence

- The Mahalanobis distance is scale independent in the following sense.
- Observations that lie on a particular ellipse all have the same Mahalanobis distance from the center of the scatter plot, even though they have different Euclidean distances.
- This feature is particularly useful when we deal with economic variables that are measured according to different scales.
- For example, inflation is measured as a percentage change whereas the unemployment rate is measured as a level.
- By multiplying $(x - \gamma)$ by the inverse of the covariance matrix, we convert all of the variables into common units.

Statistical Probabilities of Prospective Scenarios

Persistence versus Mean Reversion

- Thus far, we have considered deviations in economic conditions relative to their current values, which implicitly assumes that current conditions are likely to persist.

- Some investors may believe that mean reversion is a stronger force than persistence.
  - An economic trend has persisted for an unusually long period of time.
  - Economic conditions have deviated far from their normal range.
  - The forecasted scenarios are far into the future.

- To reflect this view, we can replace deviations from current conditions, $(x - \gamma)$, with deviations from normal conditions, $(x - \mu)$.

- More generally, investors may anchor probabilities to any chosen blend of current conditions and normal conditions, by introducing a variable θ, which is a weighted average of current and normal conditions.

$$d = (x - \theta)'\Sigma^{-1}(x - \theta)$$

$$\theta = \omega\gamma + (1 - \omega)\mu$$

- One may also use θ to express a view that economic conditions will tend towards a value other than the historical norm, for example a set of conditions associated with good or bad stock market performance.

Statistical Probabilities of Prospective Scenarios

The Mahalanobis distance is related to a scenario's probability of occurrence in the following sense.

- An observation with a high Mahalanobis distance will tend to occur less frequently than one with a low Mahalanobis distance.
- If we assume that the economic variables follow a multivariate normal distribution, we can measure the relative likelihood of events precisely.
- The likelihood of an observation decays as the Mahalanobis distance increases in accordance with an exponential function, which gives rise to the normal distribution.
- We measure the likelihood that we would observe a scenario given the chosen blend of prevailing and normal values of the economic variables used to define the scenario, together with their historical covariation, as:

$$likelihood \propto e^{-d/2}$$

d = the Mahalanobis distance

e = the base of the exponential function

$\propto$ denotes a proportionality relationship

- We then rescale the probabilities to sum to 1.

Statistical Probabilities of Prospective Scenarios

To summarize

1. We define a finite set of prospective scenarios as combinations of economic variables.

2. We calculate the Mahalanobis distances of the prospective scenarios from the prevailing and normal values of the variables used to define the scenarios.

3. We convert each prospective scenario's Mahalanobis distance into a likelihood measure.

4. We rescale the likelihoods to sum to one, which we interpret as scenario probabilities.

Case Study (as of December 2018)

Economic variables used to characterize scenarios

- Economic growth (year-over-year percent change in real GNP, seasonally adjusted)
- Unemployment rate (civilian unemployment rate, seasonally adjusted)
- Inflation (year-over-year percent change in the Consumer Price Index, seasonally adjusted)
- Interest rates (3-month Treasury rate)
- Yield curve slope (10-year Treasury rate minus 3-month Treasury rate)
- Credit spreads (Moody's BAA rate – 10-year Treasury constant maturity rate)

All economic data is sourced from the Federal Reserve Economic Data (FRED) website.

Case Study (as of December 2018)

Current and Prospective Scenarios

Economic Variables	Current	Normal	Weak	Robust
Economic Growth	3.0%	2.6%	0.9%	4.2%
Unemployment Rate	3.7%	5.3%	6.9%	3.7%
Inflation	2.2%	2.0%	0.6%	3.3%
Interest Rates	2.4%	0.1%	0.0%	2.5%
Yield Curve Slope	0.5%	2.0%	3.0%	0.5%
Credit Spreads	2.3%	2.7%	3.4%	1.9%

"Current" represents values as of December 2018. We characterize normal growth in the context of the past ten years by eliminating all six-month periods with economic growth in the top 25 percent or bottom 25 percent of the distribution and taking the median of each variable over the remaining periods. We define the weak growth scenario as a collection of one standard deviation tilts away from normal. In particular, we assume that a weak environment coincides with lower inflation, lower interest rates (bounded at zero), higher yield curve slope, higher credit spreads, lower economic growth, and a higher unemployment rate. The robust growth scenario applies the same size tilts from normal but this time in a favorable direction. We use standard deviations of the variables (relative to normal conditions) over the past 30 years.

Case Study (as of December 2018)

Scenario Probabilities

	Normal	Weak	Robust
Probability based on Current	16%	0%	84%
Probability based on Normal	75%	6%	19%
Probability based on 50/50 Blend	58%	1%	41%

We apply the equations from slides 11 and 12 to estimate the relative probability of each scenario. Covariances are based on deviations over the past 30 years (January 1989 through December 2018).

Case Study (as of December 2018)

Asset Classes

- Equities (S&P 500 Index)
- Treasury Bonds (Bloomberg U.S. Treasuries Index)
- Corporate Bonds (Bloomberg U.S. Credit Index)
- Commodities (Dow Jones Commodities Index)
- Cash Equivalents (JP Morgan 3-month Cash Index)

All market data is sourced from Thompson Reuters' Datastream.

Case Study (as of December 2018)

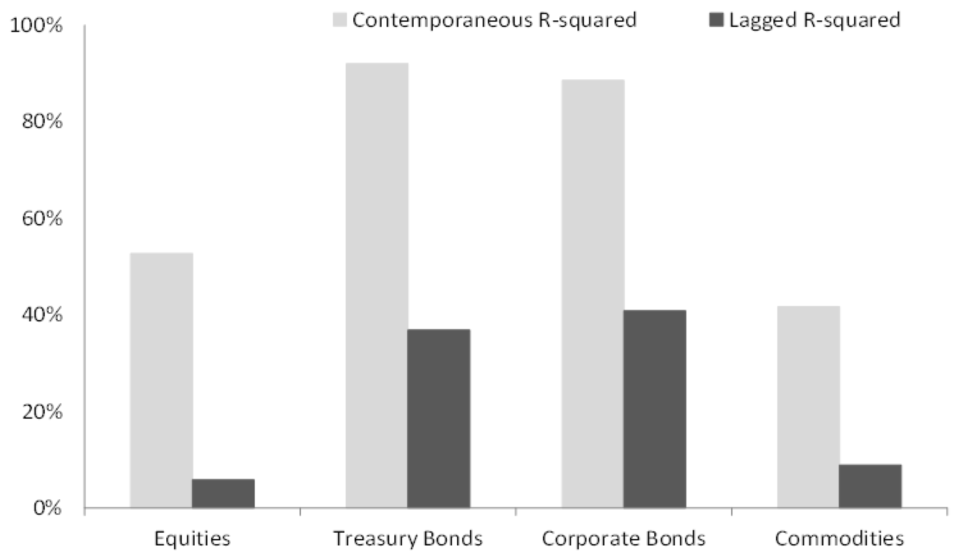
Asset Class Regression Results

	Intercept	Economic Growth	Unemployment Rate	Inflation	Interest Rates	Yield Curve Slope	Credit Spreads	R-squared
Equities	**7.4%**	0.7	**-7.7**	-1.3	-4.7	-3.6	**-13.3**	53%
p-value	**0.00**	0.23	**0.00**	0.06	0.06	0.10	**0.00**	
Treasury Bonds	**2.0%**	**0.2**	**0.4**	-0.1	**-5.2**	**-5.1**	-0.1	92%
p-value	**0.00**	**0.02**	**0.05**	0.07	**0.00**	**0.00**	0.42	
Corporate Bonds	**2.6%**	0.2	0.4	**-0.5**	**-7.3**	**-6.9**	**-6.6**	89%
p-value	**0.00**	0.07	0.16	**0.00**	**0.00**	**0.00**	**0.00**	
Commodities	4.0%	0.5	3.3	**9.9**	1.6	1.1	-1.4	42%
p-value	0.13	0.37	0.24	**0.00**	0.31	0.34	0.34	

We regress the historical returns of the asset classes on the contemporaneous historical changes of the economic variables used to define the scenarios to derive equations that we then use to predict the asset classes' returns for a given scenario. The dependent variable in each regression is the past year's return in excess of cash for the relevant asset class as of each quarter end. The independent variables are year-over-year changes in economic variables for the same periods. The p-values we report are adjusted to account for the autocorrelation induced by overlapping windows. Results are based on data beginning January 1989 and ending December 2018.

Case Study (as of December 2018)

Lagged versus Contemporaneous Regression Results



Results are based on data beginning January 1989 and ending December 2018.

Case Study (as of December 2018)

Scenario-Dependent Asset Class Returns

	Normal	Weak	Robust
Equities	-2.5%	-27.3%	13.5%
Treasury Bonds	8.9%	4.4%	3.7%
Corporate Bonds	9.1%	-1.8%	6.1%
Commodities	6.7%	-2.2%	18.5%
Cash Equivalents	2.4%	2.4%	2.4%

For each scenario, we substitute the prospective change in each economic variable into the regression equations (from slide 18) to arrive at the scenario-dependent return of each asset class. In the case of cash, we assume that the current interest rate will prevail over the next year. For all other assets we add cash to the excess returns estimated from the regressions in order to estimate expected total returns.

Case Study (as of December 2018)

Portfolio Weights

	Conservative	Moderate	Aggressive
Equities	20%	40%	60%
Treasury Bonds	30%	15%	0%
Corporate Bonds	10%	15%	20%
Commodities	10%	15%	20%
Cash Equivalents	30%	15%	0%

We consider three portfolios: conservative, moderate and aggressive.

Case Study (as of December 2018)

Scenario-Dependent Portfolio Returns

	Conservative	Moderate	Aggressive
Normal	4.4%	3.0%	1.6%
Weak	-3.8%	-10.5%	-17.2%
Robust	7.0%	10.0%	13.0%

We multiply the asset class returns in each scenario (from slide 20) by the portfolio weights (from the previous slide) to arrive at the return associated with each portfolio in each scenario.

Case Study (as of December 2018)

- Most investors are risk averse, which means that they dislike a given size loss more than they like an equal size gain.
- These preferences are typically expressed by defining utility as a concave function of portfolio return.

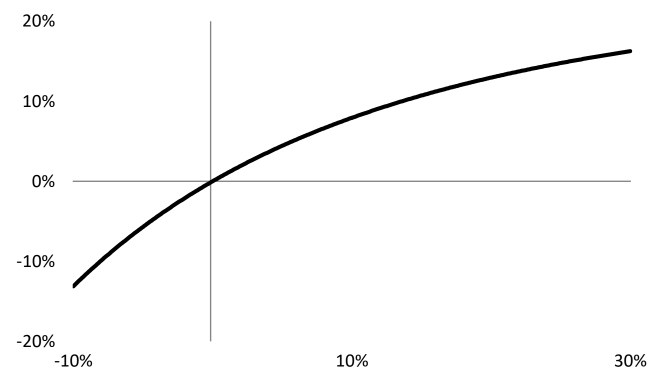
Power Utility Function

$$U_{power}(R) = \frac{(1+R)^{1-\varphi}-1}{1-\varphi}$$

U = utility

R = returns

φ = risk aversion coefficient



The exhibit depicts power utility function for returns ranging from -10% to 30%. For power utility, we set the risk aversion coefficient, φ, equal to 5.

Case Study (as of December 2018)

Scenario-Dependent Portfolio Utility

	Conservative	Moderate	Aggressive
Normal	4.0%	2.8%	1.6%
Weak	-4.2%	-14.0%	-28.2%
Robust	5.9%	7.9%	9.7%

We multiply the asset class utilities in each scenario by the portfolio weights (from slide 21) to arrive at the utility associated with each portfolio in each scenario.

Case Study (as of December 2018)

Expected Portfolio Returns and Utility

	Conservative	Moderate	Aggressive
Expected return - persistence	6.6%	8.9%	11.2%
Expected return - mean reversion	4.4%	3.5%	2.7%
Expected return - 50/50 blend	5.4%	5.7%	6.0%
Expected utility - persistence	5.6%	7.1%	8.4%
Expected utility - mean reversion	3.9%	2.8%	1.3%
Expected utility - 50/50 blend	4.7%	4.7%	4.5%

We multiply the scenario-dependent utilities (from the previous slide) by the relative probabilities of the prospect scenarios (from slide 16) to arrive at the probability-weighted expected utility of each portfolio.

Scenario Modification

How to modify scenario specifications to accord with one's views

- We cannot solve this problem analytically. Instead, we must resort to an iterative procedure to find the solution.
- We take the partial derivatives of the probability which we are targeting with respect to the scenarios and iteratively change the values of the economic variables by an arbitrarily small amount in proportion to the direction and size of the derivatives.
- We repeat this process until the rescaled probability of the targeted scenario equals our desired value.
- After doing so, we should consider how the other scenario probabilities changed to ensure that we are comfortable with all of the new probabilities.

Scenario Modification

- We first compute the gradient (set of derivatives) of the Mahalanobis distance, d_m, with respect to the vector of economic variables, x_m, for a given scenario m, to arrive at the collective set of changes that will have the largest impact on the Mahalanobis distance for that scenario.

$$\nabla d_m(x_m) = 2\Sigma^{-1}(x_m - \theta)$$

∇d_m = the gradient of the Mahalanobis distance as a function of its vector of inputs,
Σ^{-1} = the inverse of the covariance matrix of historical deviations,
θ = the vector of economic variables based on the chosen blend of current and normal conditions.

- The gradient tells us the directions and proportions to adjust each element of x_m to have the greatest impact for a given scenario, m. We repeat this calculation for every scenario.

- This is an intermediate step. Our goal is to determine the sensitivity of the rescaled probabilities to the economic variables.

Scenario Modification

- The rescaled scenario probabilities are a function of the raw probabilities, which themselves are a function of the Mahalanobis distances. We therefore have a set of nested functions.
- We use the chain rule of calculus to determine the sensitivities of these probabilities to the economic variables.
- These derivatives reveal the appropriate direction and relative size of the changes to be made in each scenario.
- We proceed iteratively until the rescaled probability equals our target.

$$\nabla p_{target}(x_m) =$$

$$\frac{M}{\sqrt{det(2\pi\Sigma)}}\left(\frac{\delta_{target}(m)}{\sum_{k=1}^{M}\xi_k} - \frac{\xi_m}{(\sum_{k=1}^{M}\xi_k)^2}\right)e^{-d_m/2}\,\Sigma^{-1}(x_m - \theta)$$

∇p_{target} is the gradient of the probability as a function of its vector of inputs,
the term $\delta_{target}(m)$ is equal to one if scenario m is the same as the scenario for which we are targeting a probability and it is equal to zero otherwise,
M is the number of economic scenarios,
N is the number of economic variables,
d_m is the Mahalanobis distance of x_m,
ξ_m is the raw probability density of d_m,
$det(\,)$ represents the determinant of a matrix.

Scenario Modification

Baseline scenarios

Economic Variables	Normal	Weak	Robust
Economic Growth	2.6%	0.9%	4.2%
Unemployment Rate	5.3%	6.9%	3.7%
Inflation	2.0%	0.6%	3.3%
Interest Rates	0.1%	0.0%	2.5%
Yield Curve Slope	2.0%	3.0%	0.5%
Credit Spreads	2.7%	3.4%	1.9%
Mahalanobis Distance	0.90	8.57	1.63
Probability based on 50/50 Blend	58%	1%	41%

Adjusted scenarios - Interim

Economic Variables	Normal		Weak		Robust	
	Adjustment*	Value	Adjustment*	Value	Adjustment*	Value
Economic Growth	0.003%	2.6%	-0.010%	0.9%	0.005%	4.3%
Unemployment Rate	0.015%	5.3%	-0.082%	6.8%	0.000%	3.7%
Inflation	0.006%	2.0%	0.030%	0.7%	0.009%	3.3%
Interest Rates	-0.010%	0.1%	-0.010%	0.0%	0.000%	2.5%
Yield Curve Slope	0.007%	2.0%	-0.109%	2.9%	-0.012%	0.4%
Credit Spreads	0.002%	2.7%	-0.131%	3.3%	-0.011%	1.9%
Mahalanobis Distance	0.03	0.92	-1.30	7.26	0.03	1.66
Probability based on 50/50 Blend	-1%	58%	1%	2%	-1%	40%

Adjusted scenarios - Final

Economic Variables	Normal		Weak		Robust	
	Adjustment*	Value	Adjustment*	Value	Adjustment*	Value
Economic Growth	0.016%	2.6%	0.001%	0.9%	0.026%	4.3%
Unemployment Rate	0.086%	5.4%	-0.404%	6.4%	0.001%	3.7%
Inflation	0.035%	2.0%	0.145%	0.8%	0.051%	3.3%
Interest Rates	-0.056%	0.1%	-0.010%	0.0%	-0.006%	2.5%
Yield Curve Slope	0.038%	2.0%	-0.444%	2.6%	-0.075%	0.4%
Credit Spreads	0.010%	2.7%	-0.412%	3.0%	-0.076%	1.9%
Mahalanobis Distance	0.16	1.06	-4.13	4.44	0.26	1.89
Probability based on 50/50 Blend	-4%	54%	9%	10%	-5%	36%

*Adjustments relative to baseline

We apply the algorithm described on slides 27 and 28 in order to increase the probability of the weak scenario to 10% (based on the 50/50 anchor).

Complex or Simple?

- Select economic variables
- Select scenarios
- Select asset classes
- Select portfolios

- Compute scenario probabilities
- Compute asset class returns
- Compute portfolio returns and utilities
- Take a break

Concluding Comments

- Some investors use scenario analysis as an alternative to mean-variance analysis because they find it more intuitive.
- However, this intuition comes at the expense of quantitative rigor, because investors rely on subjective judgment to define prospective scenarios and to assign probabilities to their occurrence.
- We introduce a new procedure that removes much of the subjectivity of scenario analysis.
- We first introduce a robust statistical procedure for determining the relative probabilities of prospective scenarios.
- We then introduce a second procedure for modifying scenarios in order to render them consistent with one's views.
- This statistically enhanced approach to scenario analysis is no doubt mathematically complex.
- But this complexity arises in the construction of the statistical process. Once this process is in place its implementation and the assessment of its output is as intuitive as any subjective approach to scenario analysis.
- Our approach differs only to the extent that it rests on a sound scientific foundation.