

“To Rebalance or Not to Rebalance?”



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To Rebalance or Not to Rebalance

It is not THE QUESTION but a very important one

- » **To rebalance – fixed-weight (FW); Not to – Buy and hold (BH)**
- » **“Passive” (inactive) versus “active”**
- » **Efficient market theory versus market inefficiency**
- » **Traditional cap-weighted indices versus alternative betas**
- » **Asset allocation FW policy versus asset-level BH benchmarks**



To Rebalance or Not to Rebalance

There have been no satisfactory answers

- » Does FW portfolios have higher returns?
 - » Is “diversification return” real or imaginary?
- » Does FW portfolios have lower risks?
- » The effects of mean-reverting or trending on portfolio rebalancing
 - » FW sells winners and buy losers (in long-only portfolios)
- » Effects of portfolio rebalancing for long-short portfolios?
- » Should we care more about terminal wealth?



To Rebalance or Not to Rebalance

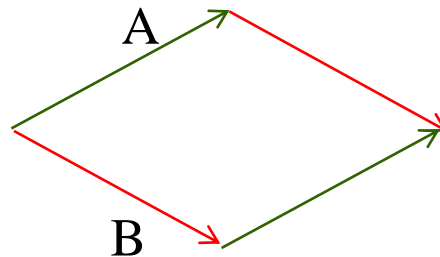
Outline

- » Direct comparison between FW and BH portfolios
- » Terminal wealth instead of average returns
- » Expected value and variance of terminal wealth – wealth Sharpe ratio
- » Long-only portfolios and long-short portfolios
- » Effects of serial correlation (a hard problem)
- » Qian, Edward, 2014, "To Rebalance or Not to Rebalance: A Statistical Analysis of Terminal Wealth of Fixed-weight and Buy-and-Hold Portfolios", available at www.ssrn.com

Rebalancing Return

A simple experiment

» Two securities A and B go up and down with zero cumulative return



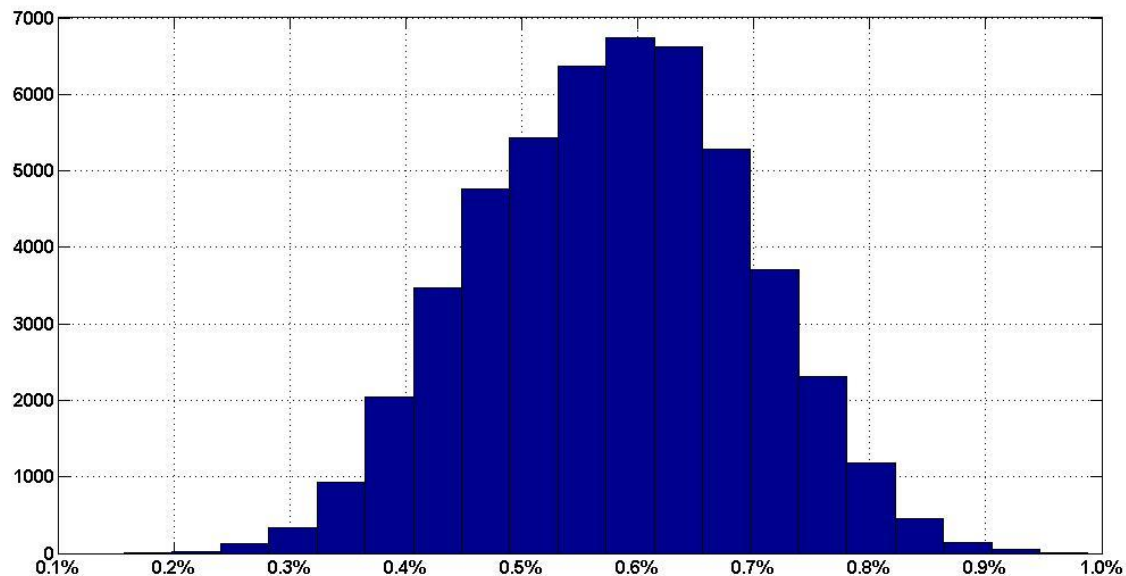
» Portfolio rebalancing generates positive return

	Year 1	Rebalance	Year 2
A \$50	\$100	\$62.5 (50%)	\$31.25
B \$50	\$25	\$62.5 (50%)	\$125
Total \$100	\$125 (25%)	\$125	\$156.25 (25%)

Rebalancing Return

A more realistic experiment

- » S&P 500 sector portfolios: rebalancing always leads to higher return
 - » Annual returns from 1990 – 2013 for 10 S&P sectors
 - » 50,000 randomly generated portfolios
 - » Alpha = annual return with rebalancing minus return with buy-and-hold





Diversification Return

Diversification “return” is not rebalancing return

» Arithmetic mean

$$\mu = \frac{1}{M} (r_1 + \dots + r_M)$$

$$g \approx \mu - \frac{1}{2} \sigma^2$$

» Geometric mean

$$1 + g = \left[(1 + r_1) \cdots (1 + r_M) \right]^{1/M}$$

$$DR = g_p - \sum_{i=1}^N w_i g_i \geq 0$$

» Diversification return is not return between two real portfolios

» $\sum_{i=1}^N w_i g_i$ IS NOT the geometric mean of the buy-and-hold portfolio

- » Qian, Edward , “Diversification Return and Leveraged Portfolios”, *The Journal of Portfolio Management*, Summer 2012, Vol. 38, No. 4: pp. 14-25



Terminal Wealth

Notations

» **M assets/N periods, return of i th asset in period n : r_{in}**

» **Expected return independent of n - return vector: $\vec{\mu}$**

$$E(r_{in}) = \mu_i, i = 1, \dots, M, n = 1, \dots, N$$

» **Covariances independent of n - covariance matrix: Σ**

$$E[(r_{in} - \mu_i)(r_{jn} - \mu_j)] = \sigma_{ij}, i, j = 1, \dots, M, n = 1, \dots, N$$

» **No serial correlation between returns of different time period**

» **Initial portfolio weights $\vec{w} = (w_1, \dots, w_N)'$**



Terminal Wealth

Notations

» Expected return of the FW portfolio

$$\mu_p = w_1\mu_1 + w_2\mu_2 + \cdots + w_M\mu_M = \sum_{i=1}^M w_i\mu_i = \vec{w}' \cdot \vec{\mu}.$$

» Volatility of the FW portfolio

$$\sigma_p^2 = \sum_{i,j=1}^M w_i w_j \sigma_{ij} = \vec{w}' \Sigma \vec{w}.$$



Terminal Wealth

Terminal wealth of \$1 investment

» FW portfolio - product of period returns

$$W_{FW} = \left(1 + \sum_{i=1}^M w_i r_{i1}\right) \cdots \left(1 + \sum_{i=1}^M w_i r_{iN}\right) = \prod_{n=1}^N \left(1 + \sum_{i=1}^M w_i r_{in}\right).$$

» BH portfolio - weighted sum of terminal wealth

$$W_{BH} = w_1(1 + r_{11}) \cdots (1 + r_{1N}) + \cdots + w_M(1 + r_{M1}) \cdots (1 + r_{MN})$$

$$W_{BH} = \sum_{i=1}^M w_i \left[\prod_{n=1}^N (1 + r_{in}) \right].$$



Terminal Wealth

Expected terminal wealth

» FW portfolio

$$E(W_{FW}) = E \left[\left(1 + \sum_{i=1}^M w_i r_{i1} \right) \cdots \left(1 + \sum_{i=1}^M w_i r_{iN} \right) \right] = (1 + \mu_p)^N$$

$$E(W_{FW}) = \left(1 + \sum_{i=1}^M w_i \mu_i \right)^N$$

» BH portfolio

$$E(W_{BH}) = w_1 E[(1 + r_{11}) \cdots (1 + r_{1N})] + \cdots + w_M E[(1 + r_{M1}) \cdots (1 + r_{MN})]$$

$$E(W_{BH}) = \sum_{i=1}^M w_i (1 + \mu_i)^N.$$



Terminal Wealth

Expected terminal wealth

» **Theorem:** for long-only portfolios, i.e., $w_i \geq 0$, $\sum_{i=1}^M w_i = 1$ the expected terminal wealth of the BH portfolio is higher than that of the FW portfolio

$$E(W_{\text{BH}}) \geq E(W_{\text{FW}})$$

» **Proof by Jensen's inequality (convex function)** $f(x) = (1 + x)^N$

$$\sum_{i=1}^M w_i (1 + \mu_i)^N \geq \left(1 + \sum_{i=1}^M w_i \mu_i \right)^N$$

» **Intuition:** don't sell winners if winners keep on winning

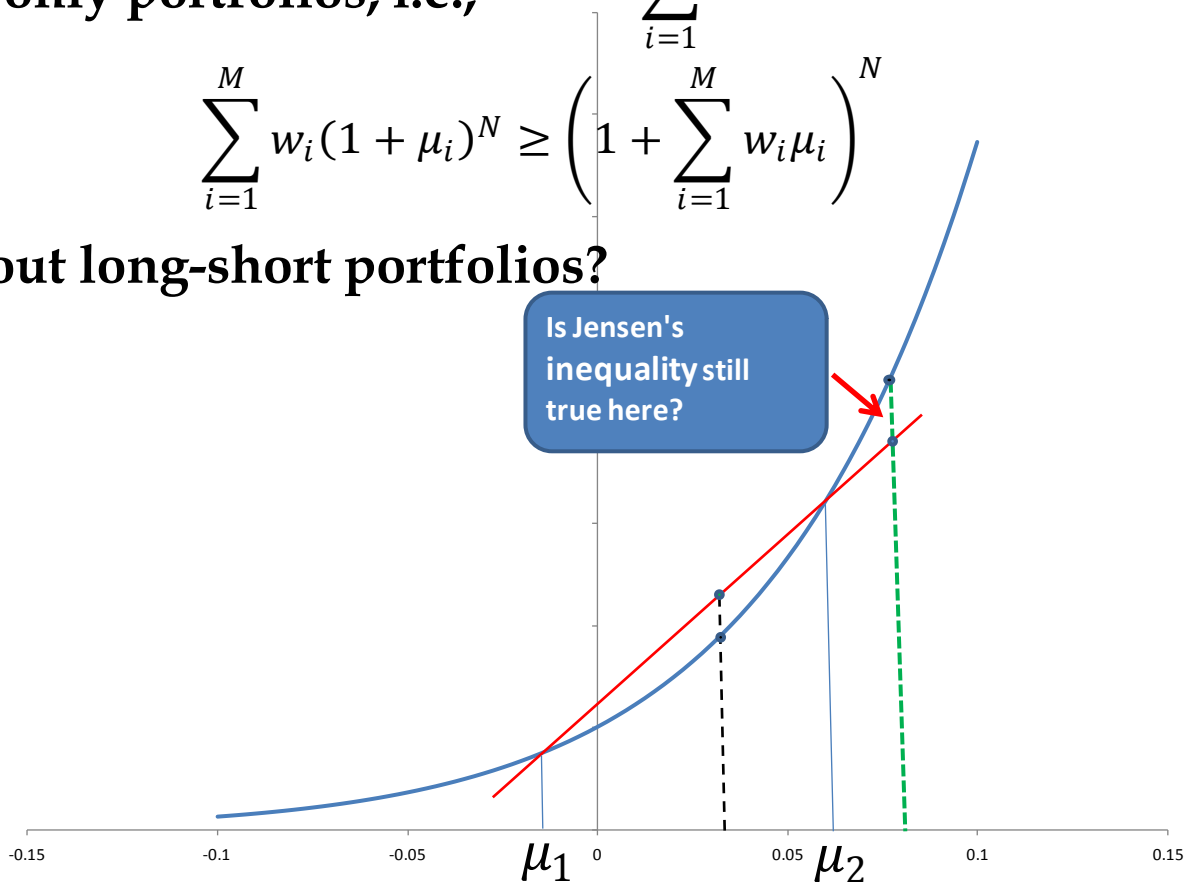
Terminal Wealth

Jensen's inequality

» For long-only portfolios, i.e., $w_i \geq 0$, $\sum_{i=1}^M w_i = 1$

$$\sum_{i=1}^M w_i (1 + \mu_i)^N \geq \left(1 + \sum_{i=1}^M w_i \mu_i \right)^N$$

» What about long-short portfolios?





Long-Short Portfolios

What about long-short portfolios?

- » Short positions: negative weights
 - » Short selling: borrow shares to sell
 - » Borrow money to buy assets
 - » Invest with derivatives (futures)

» Mathematically, we still have $\sum_{i=1}^M w_i = 1$

» Portfolio leverage if some weights are negative

$$L = \sum_{i=1}^M |w_i| > 1$$

Long-Short Portfolios

Weights of L/S portfolios

» L/S 120/20 portfolio with security A and B

» A returns 100% and B returns -50%

	Year 1	Year 1	Rebalance
A(\$120/120%)	\$240	104%	\$276 (120%) (Buy)
B(-\$20/-20%)	-\$10	-4%	-\$46 (-20%) (Sell)
Total \$100(140%)	\$230	100%(108%)	\$230

» Portfolio grows from \$100 to \$230

» Leverage decreases from 140% to 108%

» Rebalancing requires buying the winner and sell the loser

Long-Short Portfolios

Weights of L/S portfolios

» L/S 120/20 portfolio with security A and B

» A returns -50% and B returns 100%

	Year 1	Year 1	Rebalance
A(\$120/120%)	\$60	300%	\$24(120%) (Sell)
B(-\$20/-20%)	-\$40	-200%	-\$4 (-20%) (Buy)
Total \$100(140%)	\$20	100%(500%)	\$20

» Portfolio drops from \$100 to \$20

» Leverage increases from 140% to 500%!

» Rebalancing requires buying the winner and sell the loser

Long-Short Portfolios

Weights of L/S portfolios

- » When L/S portfolios have gains (losses), leverage decreases (increases)
 - » When a L/S portfolio is positioned correctly, i.e., long higher return assets and short lower return assets, its leverage decreases.
 - » When a L/S portfolio is positioned wrongly, i.e., long lower return assets and short higher return assets, its leverage increases!
- » Buy-and-hold (passive) and leverage don't mix
- » FW might perform better than BH

$$E(W_{BH}) \leq E(W_{FW})$$



Long-Short Portfolios

Expected terminal wealth

» **Theorem:** If $w_1 < 0$, $w_i \geq 0, i = 2, \dots, M$. And $\sum_{i=1}^M w_i = 1$

» In addition, $\mu_i \geq \mu_1, i = 2, \dots, M, \mu_i > \mu_1$ for some i

» Or $\mu_i \leq \mu_1, i = 2, \dots, M, \mu_i < \mu_1$ for some i ,

» then

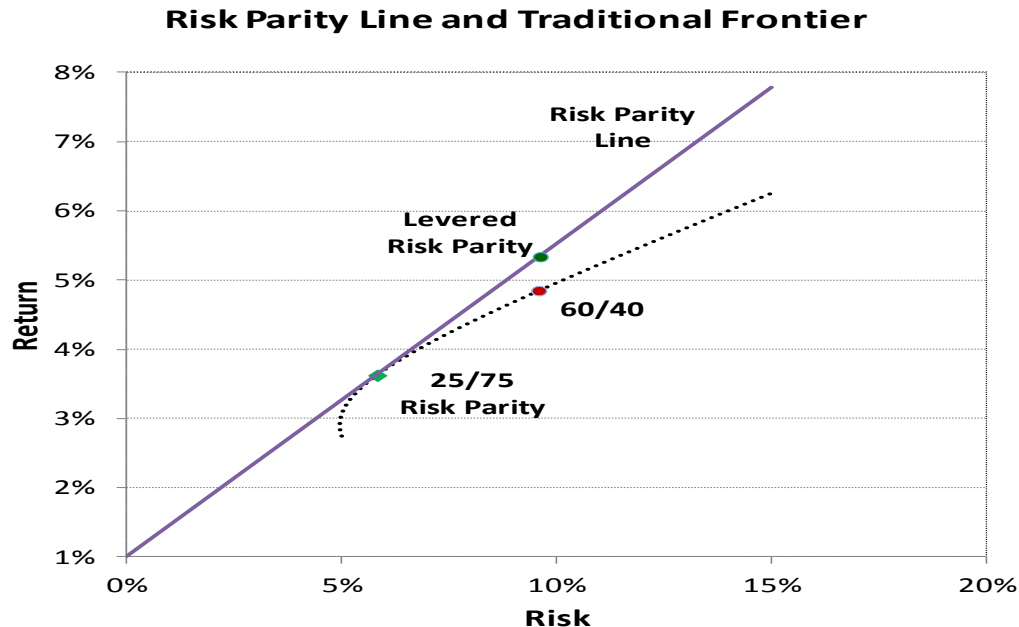
$$\sum_{i=1}^M w_i (1 + \mu_i)^N \leq \left(1 + \sum_{i=1}^M w_i \mu_i \right)^N$$

$$E(W_{BH}) \leq E(W_{FW})$$

Long-Short Portfolios

Application

- » The result can be extended to cases with more than one short assets
- » Practical application: Risk Parity portfolios
 - » Long risky assets: equity, interest rates, commodities, etc.
 - » Leveraged by shorting cash





Terminal Wealth

Expected variance

» Expected value of terminal wealth

» Long-only portfolios: $E(W_{BH}) \geq E(W_{FW})$

» Long-short portfolios $E(W_{BH}) \leq E(W_{FW})$

» But variance is also important in any investment analysis

(risk/return framework)

» What about $\text{var}(W_{BH})$ and $\text{var}(W_{FW})$?



Terminal Wealth

Expected variance

» Statistical calculation

$$\text{var}(\mathbf{x}) = E(\mathbf{x}^2) - [E(\mathbf{x})]^2.$$

» FW portfolios

$$\text{var}(W_{\text{FW}}) = \left[(1 + \mu_p)^2 + \sigma_p^2 \right]^N - (1 + \mu_p)^{2N}$$

$$\text{var}(W_{\text{FW}}) = \sum_{n=1}^N C_N^n (1 + \mu_p)^{2(N-n)} \sigma_p^{2n}$$

» BH portfolios

$$\text{var}(W_{\text{BH}}) = \sum_{i,j=1}^M w_i w_j \left[(1 + \mu_i)(1 + \mu_j) + \sigma_{ij} \right]^N - \left[\sum_{i=1}^M w_i (1 + \mu_i)^N \right]^2$$

$$\text{var}(W_{\text{BH}}) = \sum_{n=1}^N C_N^n \sum_{i,j=1}^M w_i w_j \left[(1 + \mu_i)(1 + \mu_j) \right]^{N-n} \sigma_{ij}^n.$$

Terminal Wealth

Expected variance – special case

- » Theorem: When $\mu_1 = \mu_2 = \dots = \mu_M$ and weights and covariances are non-negative
- » Then $\text{var}(W_{\text{BH}}) \geq \text{var}(W_{\text{FW}})$
- » In general, BH long-only portfolios' variance of terminal wealth is higher than that of FW portfolios.

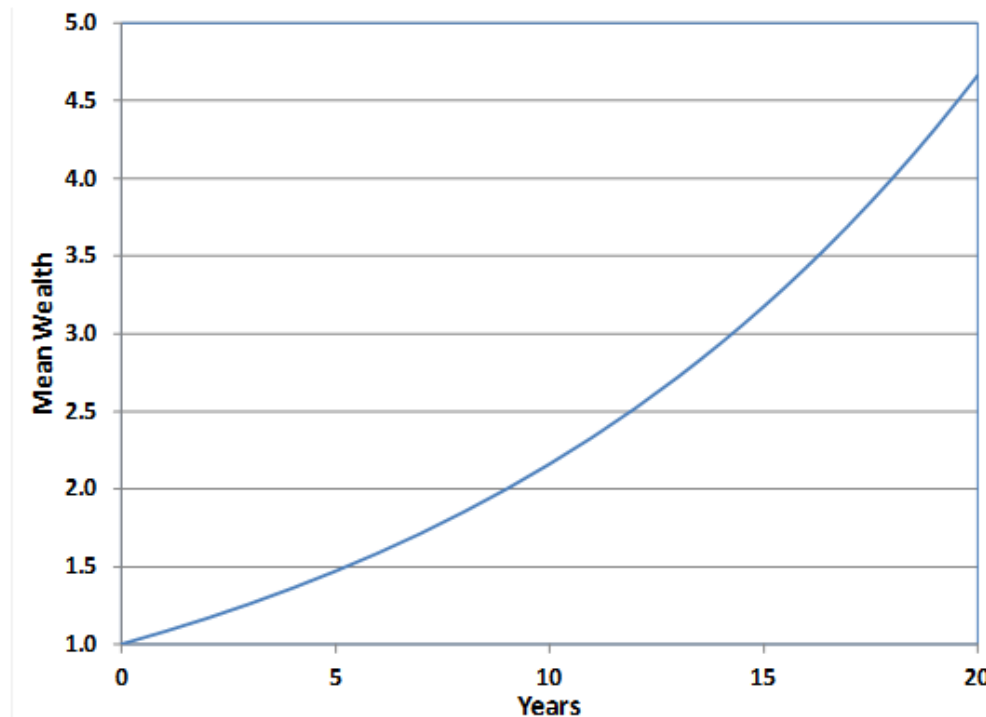


Risk-adjusted Terminal Wealth

Wealth-volatility ratio

$$\frac{E(W)}{\text{std}(W)}$$

» Example: 10 securities with equal expected return (8%), equal volatility (20%), equal pair-wise correlation (ρ); initial weight 10% each



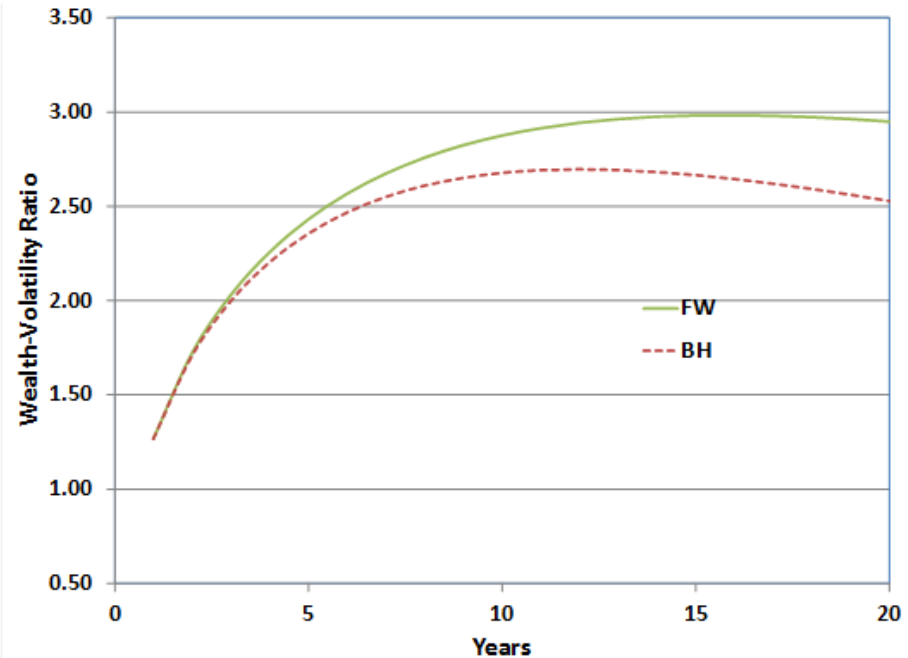
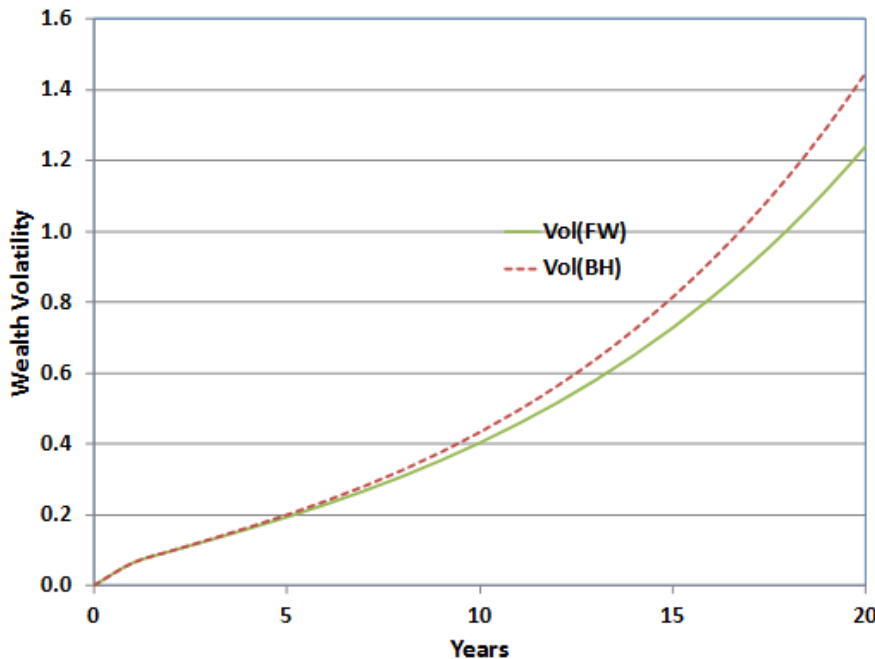


Risk-adjusted Terminal Wealth

Wealth-volatility ratio

$$\frac{E(W_{FW})}{\text{std}(W_{FW})} > \frac{E(W_{BH})}{\text{std}(W_{FW})}$$

» Example: 10 securities with equal expected return (8%), equal volatility (20%), equal pair-wise correlation ($\rho=0$)



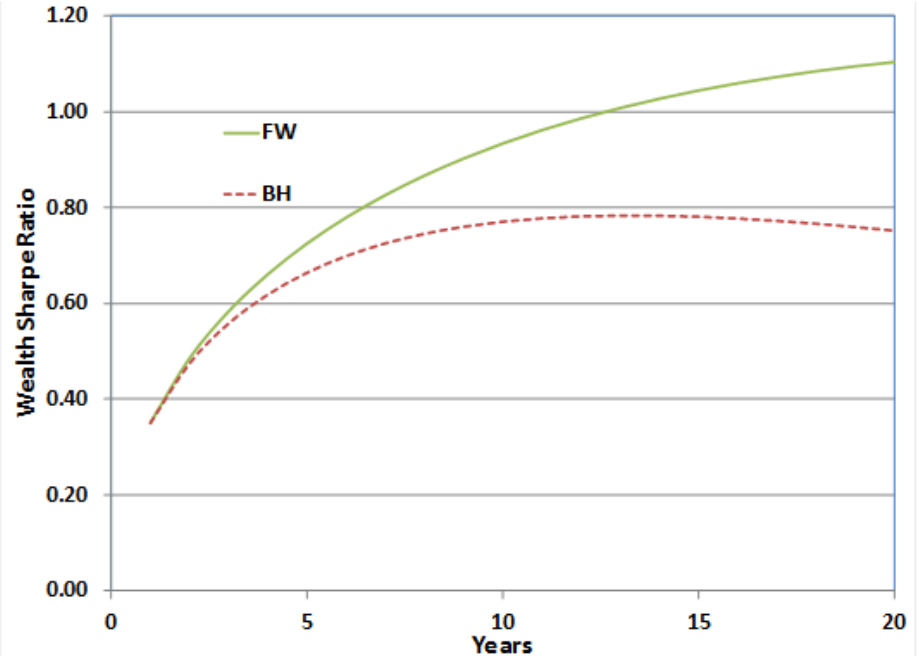
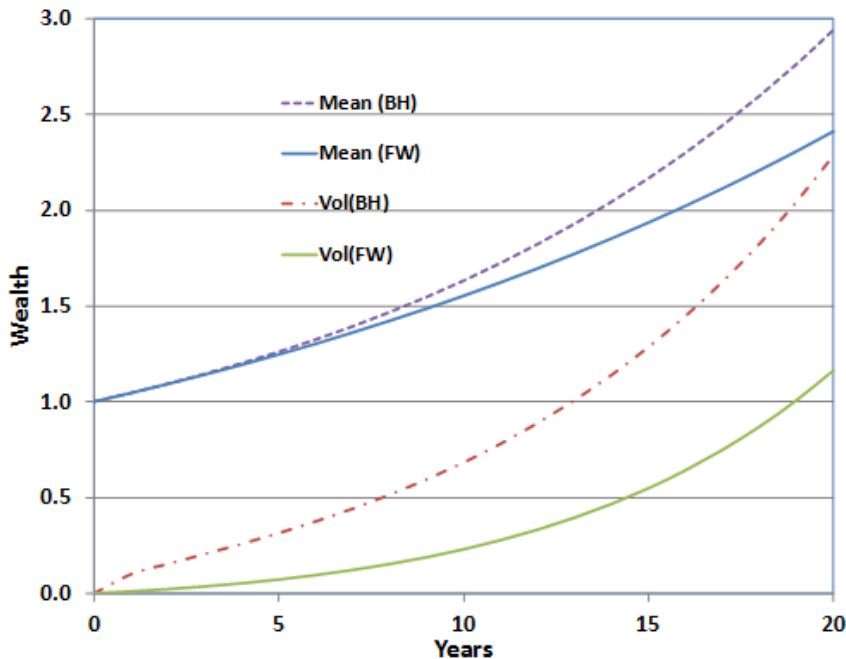


Risk-adjusted Terminal Wealth

Wealth Sharpe ratio

$$SR_W = \frac{E(W) - (1 + \mu_0)^N}{std(W)}$$

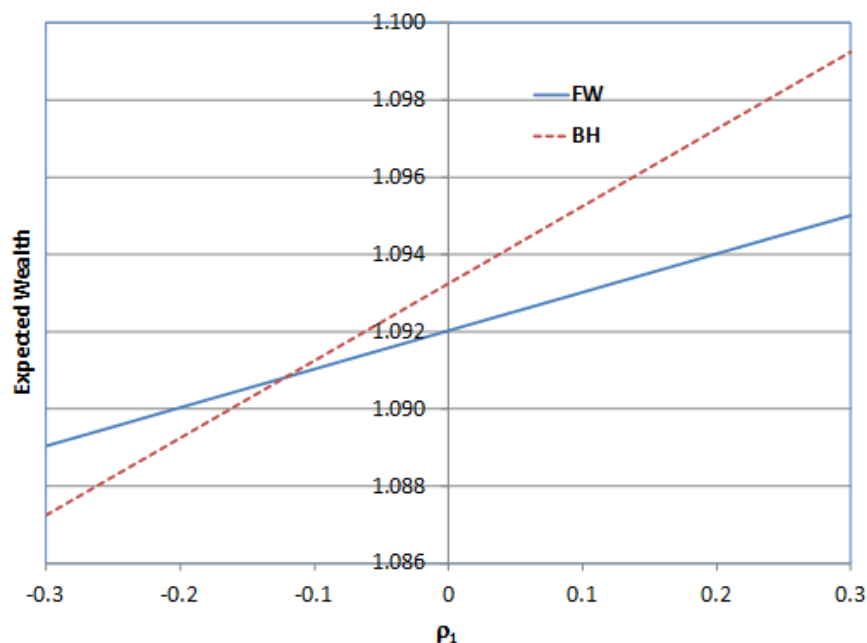
» Example: 2 assets – one risk-free with 1% return and the other 20% risk and 8% return; initial weight 50% each



Effects of Serial Correlations

Long-only portfolios

- » Mean-reverting gives FW portfolios an edge; trending or momentum gives BH portfolios an edge
- » Example: 2 assets – one risk-free with 1% return and the other 20% risk and 8% return; initial weight 50% each



$$E(W_{BH}) \leq E(W_{FW})$$

$$\text{if } \rho_1 < - \left(\frac{\mu_1 - \mu_0}{\sigma_1} \right)^2$$



Conclusions

To rebalance or not to rebalance

» Long-only portfolios

$$E(W_{BH}) \geq E(W_{FW})$$

$$\text{var}(W_{BH}) \geq \text{var}(W_{FW})$$

» FW tends to have higher risk-adjusted terminal wealth

» Long-short portfolios

$$E(W_{BH}) \leq E(W_{FW})$$

$$\text{var}(W_{BH}) ? \text{var}(W_{FW})$$

» Buy-and-hold and leveraged portfolio is not a good combination

» Serial correlation

» Mean-reverting is beneficial to FW long-only portfolios; trending is beneficial to BH long-only portfolios

» For long-short portfolios, times series trending and cross-sectional reversal is the best.



Conclusions

Practical implications – “To Rebalance!”

- » Investors have fixed-weight asset allocation portfolios but buy-and-hold indices
 - » The practice is rather inconsistent
- » Capitalization-weighted indices are BH and they often underperformed naïve equally-weighted portfolio and other kinds of alternative indices
 - » Cap-weighted indices are not diversified (not passive)
 - » Cap-weighted indices are not rebalanced
- » Long-short portfolios must be rebalanced regularly



Appendix

Diversification return

» Arithmetic mean

$$\mu = \frac{1}{M} (r_1 + \dots + r_M)$$

$$g \approx \mu - \frac{1}{2} \sigma^2$$

» Geometric mean

$$1 + g = \left[(1 + r_1) \cdot \dots \cdot (1 + r_M) \right]^{1/M}$$

$$g_p = \mu_p - \frac{1}{2} \sigma_p^2 = \sum_{i=1}^N w_i \mu_i - \frac{1}{2} \sigma_p^2 = \sum_{i=1}^N w_i \left(g_i + \frac{1}{2} \sigma_i^2 \right) - \frac{1}{2} \sigma_p^2$$

$$g_p - \sum_{i=1}^N w_i g_i = \frac{1}{2} \left[\sum_{i=1}^N w_i \sigma_i^2 - \sigma_p^2 \right] \geq 0$$