Beyond Risk Parity: Using Non-Gaussian Risk Measures and Risk Factors

Boston QWAFAFEW

Thierry Roncalli* and Guillaume Weisang†

*Lyxor Asset Management, France
†Clark University, Worcester, MA, USA

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Outline

Risk Parity with Non-Gaussian Risk Measures
  Risk Allocation Principle
  Convex Risk Measures
  Risk Budgeting with Convex Risk Measures

Risk Parity Portfolios with Risk Factors
  Motivations
  Risk Decomposition with Risk Factors
  Risk Budgeting

Applications
  Some Famous Risk Factor Models
  Diversifying a Portfolio of Hedge Funds
  Strategic Asset Allocation
Risk allocation

- How to allocate risk in a fair and effective way?

- It requires coherent and convex risk measures $\mathcal{R}(x)$ (Artzner et al., 1999; Föllmer and Schied, 2002).

Subadditivity
Homogeneity
Convexity
Monotonicity
Translation invariance

- It must satisfy some properties (Kalkbrener, 2005; Tasche, 2008).

  - Full allocation
  - RAPM compatible
  - Diversification compatible
Let $\Pi = \sum_{i=1}^{n} \Pi_i$ be the P&L of the portfolio. The risk-adjusted performance measure (RAPM) is defined by

$$\text{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathcal{R}(\Pi)}$$

and the portfolio-related RAPM of the i-th asset

$$\text{RAPM}(\Pi_i | \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathcal{R}(\Pi_i | \Pi)}$$
Risk Allocation with Respect to P&L

Let $\Pi = \sum_{i=1}^{n} \Pi_i$ be the P&L of the portfolio. The risk-adjusted performance measure (RAPM) is defined by

$$\text{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathcal{R}(\Pi)}$$

and the portfolio-related RAPM of the i-th asset

$$\text{RAPM}(\Pi_i | \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathcal{R}(\Pi_i | \Pi)}$$

From an economic point of view, $\mathcal{R}(\Pi_i | \Pi)$ must satisfy two properties

1. Risk contributions $\mathcal{R}(\Pi_i | \Pi)$ satisfy the full allocation property if

$$\sum_{i=1}^{n} \mathcal{R}(\Pi_i | \Pi) = \mathcal{R}(\Pi)$$

2. They are RAPM compatible if there are some $\varepsilon_i > 0$ such that:

$$\text{RAPM}(\Pi_i | \Pi) > \text{RAPM}(\Pi) \Rightarrow \text{RAPM}(\Pi + h\Pi_i) > \text{RAPM}(\Pi)$$

for all $0 < h < \varepsilon_i$. 

T. Roncalli & G. Weisang (Lyxor & ClarkU)

Beyond Risk Parity

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Risk Allocation with Respect to Portfolio Weights

In this case, Tasche (2008) shows that

\[ \mathcal{R}(\Pi_i | \Pi) = \frac{d}{dh} \mathcal{R}(\Pi + h\Pi_i) \bigg|_{h=0} \]

if, and only if, \( \mathcal{R}(\Pi) \) is homogeneous of degree 1.

Within the previous framework, we obtain the risk contribution \( RC_i \) of asset \( i \) as

\[ RC_i = x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} \]

and the risk measure satisfies the Euler decomposition

\[ \mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} = \sum_{i=1}^{n} RC_i \]
Some Examples

Let \( L(x) \) be the loss of the portfolio \( x \).

- The volatility of the loss
  \[
  \sigma(L(x)) = \sigma(x)
  \]

- The standard deviation based risk measure
  \[
  \text{SD}_c(x) = -\mu(x) + c \cdot \sigma(x)
  \]

- The value-at-risk
  \[
  \text{VaR}_\alpha(x) = \inf \{ \ell : \Pr \{ L \leq \ell \} \geq \alpha \} = F^{-1}(\alpha)
  \]

- The expected shortfall
  \[
  \text{ES}_\alpha(x) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(x) \, du
  = \mathbb{E}[L(x) \mid L(x) \geq \text{VaR}_\alpha(x)]
  \]

Gaussian case

Volatility, value-at-risk and expected shortfall are equivalent.
Non-Gaussian Risk Measures

For the value-at-risk, Gourieroux et al. (2000) shows that

\[ RC_i = \mathbb{E}[L_i | L = \text{VaR}_\alpha (L)] \]

whereas we have for the expected shortfall (Tasche, 2002)

\[ RC_i = \mathbb{E}[L_i | L \geq \text{VaR}_\alpha (L)] \]

Example

EW portfolio with 2 assets (Clayton copula + student’s t margins)\(^2\)

<table>
<thead>
<tr>
<th></th>
<th>Vol</th>
<th>VAR</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(x) )</td>
<td>24.51</td>
<td>18.32</td>
<td>35.99</td>
</tr>
<tr>
<td>( RC_1(x) )</td>
<td>36.5%</td>
<td>34.2%</td>
<td>35.2%</td>
</tr>
<tr>
<td>( RC_2(x) )</td>
<td>63.5%</td>
<td>65.8%</td>
<td>64.8%</td>
</tr>
</tbody>
</table>

\(^2\)see Roncalli (2012).
Non-Gaussian Risk Contributions

1. Value-at-risk with elliptical distributions (Carroll et al., 2001)

\[ RC_i = \mathbb{E}[L_i] + \frac{\text{cov}(L, L_i)}{\sigma^2(L)} (\text{VaR}_\alpha(L) - \mathbb{E}[L]) \]

2. Historical value-at-risk with non-elliptical distributions

\[ RC_i = \text{VaR}_\alpha(L) \frac{\sum_{j=1}^{m} \mathcal{K}(L^{(j)} - \text{VaR}_\alpha(L)) L^{(j)}_i}{\sum_{j=1}^{m} \mathcal{K}(L^{(j)} - \text{VaR}_\alpha(L)) L^{(j)}} \]

where \( \mathcal{K}(u) \) is a kernel function (Epperlein and Smillie, 2006).


\[ \text{VaR}_\alpha(L) = -x^\top \mu + z \cdot \sqrt{x^\top \Sigma x} \]

where

\[ z = z_\alpha + \frac{1}{6} \left( z_\alpha^2 - 1 \right) \gamma_1 + \frac{1}{24} \left( z_\alpha^3 - 3z_\alpha \right) \gamma_2 - \frac{1}{36} \left( 2z_\alpha^3 - 5z_\alpha \right) \gamma_1^2 \]

with \( z_\alpha = \Phi^{-1}(\alpha) \), \( \gamma_1 \) is the skewness and \( \gamma_2 \) is the excess kurtosis\(^3\).

\(^3\)See Roncalli (2012) for the detailed formula of the risk contribution.
Properties of RB Portfolios

Let’s consider the \textbf{long-only} RB portfolio defined by

\[ RC_i = b_i \mathcal{R}(x) \]

where \( b_i \) is the risk budget assigned to the \( i^{\text{th}} \) asset.

Bruder and Roncalli (2012) show that:

- The RB portfolio exists if \( b_i \geq 0 \);
- The RB portfolio is unique if \( b_i > 0 \);
- The risk measure of the RB portfolio is located between those of the minimum risk portfolio and the weight budgeting portfolio:

\[
\mathcal{R}(x_{\text{mr}}) \leq \mathcal{R}(x_{\text{rb}}) \leq \mathcal{R}(x_{\text{wb}})
\]

- If the RB portfolio is optimal\(^4\), the performance contributions are equal to the risk contributions.

\(^4\)In the sense of mean-quadratic risk utility function.
An Example of RB Portfolio

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.00%</td>
<td>29.40%</td>
<td>14.70%</td>
<td>70.43%</td>
</tr>
<tr>
<td>2</td>
<td>20.00%</td>
<td>16.63%</td>
<td>3.33%</td>
<td>15.93%</td>
</tr>
<tr>
<td>3</td>
<td>30.00%</td>
<td>9.49%</td>
<td>2.85%</td>
<td>13.64%</td>
</tr>
</tbody>
</table>

Volatility: 20.87%

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.15%</td>
<td>28.08%</td>
<td>8.74%</td>
<td>50.00%</td>
</tr>
<tr>
<td>2</td>
<td>21.90%</td>
<td>15.97%</td>
<td>3.50%</td>
<td>20.00%</td>
</tr>
<tr>
<td>3</td>
<td>46.96%</td>
<td>11.17%</td>
<td>5.25%</td>
<td>30.00%</td>
</tr>
</tbody>
</table>

Volatility: 17.49%

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.69%</td>
<td>27.31%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
<tr>
<td>2</td>
<td>32.44%</td>
<td>16.57%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
<tr>
<td>3</td>
<td>47.87%</td>
<td>11.23%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
</tbody>
</table>

Volatility: 16.13%
Of the Importance of the Asset Universe

Example with 4 assets

- We assume equal volatilities and uniform correlations $\rho$.
- The ERC portfolio is the EW portfolio $x_1^{(4)} = x_2^{(4)} = x_3^{(4)} = x_4^{(4)} = 25\%$.
- We add a fifth asset which is perfectly correlated to the fourth asset.
- If $\rho = 0$, the ERC portfolio becomes $x_1^{(5)} = x_2^{(5)} = x_3^{(5)} = 22.65\%$ and $x_4^{(5)} = x_5^{(5)} = 16.02\%$.
- We would prefer the allocation to be $x_1^{(5)} = x_2^{(5)} = x_3^{(5)} = 25\%$ and $x_4^{(5)} = x_5^{(5)} = 12.5\%$. 

Figure: 4 assets versus 5 assets
Which Risk Would You Like to Diversify?

- $m$ primary assets ($A'_1, \ldots, A'_m$) with a covariance matrix $\Omega$.
- $n$ synthetic assets ($A_1, \ldots, A_n$) which are composed of the primary assets.
- $W = (w_{i,j})$ is the weight matrix such that $w_{i,j}$ is the weight of the primary asset $A'_j$ in the synthetic asset $A_i$.

Example

- 6 primary assets and 3 synthetic assets.
- The volatilities of these assets are respectively 20%, 30%, 25%, 15%, 10% and 30%. We assume that the primary assets are not correlated.
- We consider three equally-weighted synthetic assets with

$$W = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{2} & & \\
\end{pmatrix}$$
### Which Risk Would You Like to Diversify?

**Risk Decomposition of Portfolio #1**

#### Along synthetic assets $A_1, \ldots, A_n$

<table>
<thead>
<tr>
<th>$\sigma(x)$ = 10.19%</th>
<th>$x_i$</th>
<th>MR ($A_i$)</th>
<th>RC ($A_i$)</th>
<th>RC* ($A_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>36.00%</td>
<td>9.44%</td>
<td>3.40%</td>
<td>33.33%</td>
</tr>
<tr>
<td>$A_2$</td>
<td>38.00%</td>
<td>8.90%</td>
<td>3.38%</td>
<td>33.17%</td>
</tr>
<tr>
<td>$A_3$</td>
<td>26.00%</td>
<td>13.13%</td>
<td>3.41%</td>
<td>33.50%</td>
</tr>
</tbody>
</table>

#### Along primary assets $A'_1, \ldots, A'_m$

<table>
<thead>
<tr>
<th>$\sigma(y)$ = 10.19%</th>
<th>$y_i$</th>
<th>MR ($A'_i$)</th>
<th>RC ($A'_i$)</th>
<th>RC* ($A'_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A'_1$</td>
<td>9.00%</td>
<td>3.53%</td>
<td>0.32%</td>
<td>3.12%</td>
</tr>
<tr>
<td>$A'_2$</td>
<td>9.00%</td>
<td>7.95%</td>
<td>0.72%</td>
<td>7.02%</td>
</tr>
<tr>
<td>$A'_3$</td>
<td>31.50%</td>
<td>19.31%</td>
<td>6.08%</td>
<td>59.69%</td>
</tr>
<tr>
<td>$A'_4$</td>
<td>31.50%</td>
<td>6.95%</td>
<td>2.19%</td>
<td>21.49%</td>
</tr>
<tr>
<td>$A'_5$</td>
<td>9.50%</td>
<td>0.93%</td>
<td>0.09%</td>
<td>0.87%</td>
</tr>
<tr>
<td>$A'_6$</td>
<td>9.50%</td>
<td>8.39%</td>
<td>0.80%</td>
<td>7.82%</td>
</tr>
</tbody>
</table>

$\Rightarrow$ The portfolio seems well diversified on synthetic assets, but 80% of the risk is on assets 3 and 4.
Which Risk Would You Like to Diversify?

Risk Decomposition of Portfolio #2

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(x) = 9.47%$</th>
<th>$x_i$</th>
<th>$\text{MR}(A_i)$</th>
<th>$\text{RC}(A_i)$</th>
<th>$\text{RC}^*(A_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>48.00%</td>
<td>9.84%</td>
<td>4.73%</td>
<td>49.91%</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>50.00%</td>
<td>9.03%</td>
<td>4.51%</td>
<td>47.67%</td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>2.00%</td>
<td>11.45%</td>
<td>0.23%</td>
<td>2.42%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(y) = 9.47%$</th>
<th>$y_i$</th>
<th>$\text{MR}(A'_i)$</th>
<th>$\text{RC}(A'_i)$</th>
<th>$\text{RC}^*(A'_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A'_1$</td>
<td>12.00%</td>
<td>5.07%</td>
<td>0.61%</td>
<td>6.43%</td>
<td></td>
</tr>
<tr>
<td>$A'_2$</td>
<td>12.00%</td>
<td>11.41%</td>
<td>1.37%</td>
<td>14.46%</td>
<td></td>
</tr>
<tr>
<td>$A'_3$</td>
<td>25.50%</td>
<td>16.84%</td>
<td>4.29%</td>
<td>45.35%</td>
<td></td>
</tr>
<tr>
<td>$A'_4$</td>
<td>25.50%</td>
<td>6.06%</td>
<td>1.55%</td>
<td>16.33%</td>
<td></td>
</tr>
<tr>
<td>$A'_5$</td>
<td>12.50%</td>
<td>1.32%</td>
<td>0.17%</td>
<td>1.74%</td>
<td></td>
</tr>
<tr>
<td>$A'_6$</td>
<td>12.50%</td>
<td>11.88%</td>
<td>1.49%</td>
<td>15.69%</td>
<td></td>
</tr>
</tbody>
</table>

⇒ This portfolio is more diversified than the previous portfolio if we consider primary assets.
The Factor Model

- $n$ assets \( \{A_1, \ldots, A_n\} \) and $m$ risk factors \( \{F_1, \ldots, F_m\} \).
- \( R_t \) is the \((n \times 1)\) vector of asset returns at time $t$ and $\Sigma$ its associated covariance matrix.
- \( F_t \) is the \((m \times 1)\) vector of factor returns at $t$ and $\Omega$ its associated covariance matrix.
- We assume the following linear factor model
  \[
  R_t = A F_t + \varepsilon_t
  \]
  with \( F_t \) and $\varepsilon_t$ two uncorrelated random vectors. The covariance matrix of $\varepsilon_t$ is noted $D$. We have:
  \[
  \Sigma = A \Omega A^\top + D
  \]
- The P&L of the portfolio $x$ is
  \[
  \Pi_t = x^\top R_t = x^\top A F_t + x^\top \varepsilon_t = y^\top F_t + \eta_t
  \]
  with $y = A^\top x$ and $\eta_t = x^\top \varepsilon_t$. 
First Route to Risk Decomposition

Let $B = A^\top$ and $B^+$ the Moore-Penrose inverse of $B$. We have therefore

$$x = B^+ y + e$$

where $e = (I_n - B^+ B) x$ is a $(n \times 1)$ vector in the kernel of $B$.

We consider a convex risk measure $\mathcal{R}(x)$. We have

$$\frac{\partial \mathcal{R}(x)}{\partial x_i} = \left( \frac{\partial \mathcal{R}(y, e)}{\partial y} B \right)_i + \left( \frac{\partial \mathcal{R}(y, e)}{\partial e} (I_n - B^+ B) \right)_i$$

Decomposition of the risk by $m$ common factors and $n$ idiosyncratic factors $\Rightarrow$ **Identification problem!**
Second Route to Risk Decomposition

Meucci (2007) considers the following decomposition

\[ x = \begin{pmatrix} B^+ & \tilde{B}^+ \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \tilde{B}^\top \tilde{y} \]

where \( \tilde{B}^+ \) is any \( n \times (n - m) \) matrix that spans the left nullspace of \( B^+ \).

Decomposition of the risk by \( m \) common factors and \( n - m \) residual factors \( \Rightarrow \) **Better identified problem.**
Euler Decomposition of the Risk Measure

**Theorem**

The risk contributions of common and residual risk factors are

\[
\text{RC} (\mathcal{F}_j) = \left( A^\top x \right)_j \cdot \left( A + \frac{\partial \mathcal{R}(x)}{\partial x} \right)_j
\]

\[
\text{RC} (\tilde{\mathcal{F}}_j) = \left( \tilde{B}x \right)_j \cdot \left( \tilde{B} \frac{\partial \mathcal{R}(x)}{\partial x} \right)_j
\]

They satisfy the Euler allocation principle

\[
\sum_{j=1}^{m} \text{RC} (\mathcal{F}_j) + \sum_{j=1}^{n-m} \text{RC} (\tilde{\mathcal{F}}_j) = \mathcal{R}(x)
\]

⇒ Risk contributions of the risk factors (resp. to assets) are related to the marginal risks of assets (resp. of risk factors).

⇒ The main important quantity is marginal risk, not risk contribution!
An Example

- We consider 4 assets and 3 factors.
- The loadings matrix is
  \[
  A = \begin{pmatrix}
  0.9 & 0 & 0.5 \\
  1.1 & 0.5 & 0 \\
  1.2 & 0.3 & 0.2 \\
  0.8 & 0.1 & 0.7 \\
  \end{pmatrix}
  \]
- The three factors are uncorrelated and their volatilities are equal to 20%, 10% and 10%.
- We consider a diagonal matrix \( D \) with specific volatilities 10%, 15%, 10% and 15%.
Beta Contribution vs. Risk Contribution

The linear model is

\[
\begin{pmatrix}
R_{1,t} \\
R_{2,t} \\
R_{3,t}
\end{pmatrix} =
\begin{pmatrix}
0.9 & 0.7 \\
0.3 & 0.5 \\
0.8 & -0.2
\end{pmatrix}
\begin{pmatrix}
F_{1,t} \\
F_{2,t}
\end{pmatrix} +
\begin{pmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t} \\
\varepsilon_{3,t}
\end{pmatrix}
\]

The factor volatilities are equal to 10% and 30%, while the idiosyncratic volatilities are equal to 3%, 5% and 2%.

If we consider the volatility risk measure, we obtain

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>(1/3, 1/3, 1/3)</th>
<th>(7/10, 7/10, −4/10)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factor</strong></td>
<td><strong>β</strong></td>
<td><strong>RC</strong></td>
</tr>
<tr>
<td>(F_1)</td>
<td>0.67</td>
<td>31%</td>
</tr>
<tr>
<td>(F_2)</td>
<td>0.33</td>
<td>69%</td>
</tr>
</tbody>
</table>

The first portfolio has a bigger beta in factor 1 than in factor 2, but about 70% of its risk is explained by the second factor. For the second portfolio, the risk w.r.t. the first factor is very small even if its beta is significant.
Matching the Risk Budgets

We consider the risk budgeting problem $\text{RC} \left( F_j \right) = b_j \mathcal{R} \left( x \right)$. It can be formulated as a quadratic problem as in Bruder and Roncalli (2012):

$$ (y^*, \tilde{y}^*) = \arg \min_{y, \tilde{y}} \sum_{j=1}^{m} \left( \text{RC} \left( F_j \right) - b_j \mathcal{R} \left( y, \tilde{y} \right) \right)^2 $$

u.c. \quad \begin{cases} 1^\top x = 1 \\ 0 \preceq x \preceq 1 \end{cases}

This problem is tricky because the first order conditions are PDE!

Some special cases

- Positive factor weights ($y \geq 0$) with $m = n \Rightarrow$ a unique solution.
- Positive factor weights ($y \geq 0$) with $m < n \Rightarrow$ at least one solution.
- Positive asset weights ($x \geq 0$ or long-only portfolio) $\Rightarrow$ zero, one or more solutions.
The Separation Principle

The problem is unconstrained with respect to the residual factors \( \tilde{F}_t \Rightarrow \) we can solve the problem in two steps

1. The first problem is \( \tilde{R}(y) = \inf \tilde{y} \ R(y, \tilde{y}) \) and we obtain \( \tilde{y} = \varphi(y) \);
2. The second problem is \( y^* = \arg \min \tilde{R}(y) \).

The solution is then given by:

\[
x^* = B^+ y^* + \tilde{B}^+ \varphi(y^*)
\]
The Separation Principle
Application to the Volatility Risk Measure

We have
\[ \bar{\Omega} = \text{cov} (\mathcal{F}_t, \tilde{\mathcal{F}}_t) = \begin{pmatrix} \Omega & \Gamma^\top \\ \Gamma & \tilde{\Omega} \end{pmatrix} \]

The expression of the risk measure becomes
\[ R(y, \tilde{y}) = \tilde{y}^\top \bar{\Omega} \tilde{y} = y^\top \Omega y + \tilde{y}^\top \tilde{\Omega} \tilde{y} + 2\tilde{y}^\top \Gamma^\top y \]

We obtain \( \tilde{y} = \varphi(y) = -\bar{\Omega}^{-1} \Gamma^\top y \) and the problem is thus reduced to \( y^* = \arg\min y^\top Sy \) with \( S = \Omega - \Gamma \bar{\Omega}^{-1} \Gamma^\top \) the Schur complement of \( \bar{\Omega} \). Because we have \( \Gamma^\top = (B^+)^\top \Sigma \tilde{B}^+ \), we obtain:
\[ x^* = B^+ y^* + \tilde{B}^+ \varphi(y^*) = \left( B^+ - \tilde{B}^+ \bar{\Omega}^{-1} (B^+)^\top \Sigma \tilde{B}^+ \right) y^* \]

**Remark**

If \( \mathcal{F}_t \) and \( \tilde{\mathcal{F}}_t \) are uncorrelated (e.g. PCA factors), a solution of the form \( (y^*, 0) \) exists and the (un-normalized) solution is given by \( x^* = B^+ y^* \).
The Separation Principle

Adding Long-Only Constraints

If we want to consider long-only allocations $x$, we must also include the following constraint

$$ x = B^+ y + \tilde{B}^+ \tilde{y} \succeq 0 $$

- The solution may not exist even if $\varphi$ is convex.
- The existence of the solution implies that there exists $\lambda = (\lambda_x, \lambda_y) \succeq 0$ such that:

$$ \left( A^+ - (\tilde{B}^+)^T \Sigma \tilde{\Omega}^{-1} (\tilde{B}^+)^T \right) \lambda_x + \lambda_y = 0 $$

We may show that this condition is likely to be verified for some non trivial $\lambda \in \mathbb{R}^{n+m}_+$. In such case, there exists $\zeta > 0$ such that $0 \leq \min y_j \leq \zeta$.

$\Rightarrow$ interpretation of this result with the convexity factor of the yield curve.
Matching the Risk Budgets

An Example (Slide 19)

If \( b = (49\%, 25\%, 25\%) \), \( x^* = (15.1\%, 39.4\%, 0.9\%, 45.6\%) \). ⇒ It is a long-only portfolio.

### Matching the risk budgets

\( b = (19\%, 40\%, 40\%) \)

<table>
<thead>
<tr>
<th>( F_i )</th>
<th>( y_i )</th>
<th>( \text{RC}(F_i) )</th>
<th>( \text{RC}^*(F_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>92.90%</td>
<td>4.45%</td>
<td>19.00%</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>28.55%</td>
<td>9.36%</td>
<td>40.00%</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>45.21%</td>
<td>9.36%</td>
<td>40.00%</td>
</tr>
<tr>
<td>( \bar{F}_1 )</td>
<td>-23.57%</td>
<td>0.23%</td>
<td>1.00%</td>
</tr>
</tbody>
</table>

\( \sigma(y) = 23.41\% \)

### Imposing the long-only constraint with

\( b = (19\%, 40\%, 40\%) \)

<table>
<thead>
<tr>
<th>( F_i )</th>
<th>( y_i )</th>
<th>( \text{RC}(F_i) )</th>
<th>( \text{RC}^*(F_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>89.85%</td>
<td>6.19%</td>
<td>28.37%</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>23.13%</td>
<td>6.63%</td>
<td>30.40%</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>47.02%</td>
<td>8.99%</td>
<td>41.20%</td>
</tr>
<tr>
<td>( \bar{F}_1 )</td>
<td>2.53%</td>
<td>0.01%</td>
<td>0.03%</td>
</tr>
</tbody>
</table>

\( \sigma(y) = 21.82\% \)

### Corresponding portfolio \( x^* \)

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>( x_i )</th>
<th>( \text{RC}_i )</th>
<th>( \text{RC}^*_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>-26.19%</td>
<td>-3.70%</td>
<td>-15.81%</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>32.69%</td>
<td>6.94%</td>
<td>29.63%</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>14.28%</td>
<td>2.91%</td>
<td>12.45%</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>79.22%</td>
<td>17.26%</td>
<td>73.73%</td>
</tr>
<tr>
<td>( \bar{A}_1 )</td>
<td>-3.70%</td>
<td>-15.81%</td>
<td></td>
</tr>
<tr>
<td>( \bar{A}_2 )</td>
<td>-15.81%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \bar{A}_3 )</td>
<td>-12.45%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \bar{A}_4 )</td>
<td>-73.73%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \sigma(x) = 23.41\% \)
Concentration index

Let \( p \in \mathbb{R}_+^n \) such that \( \mathbf{1}^\top p = 1 \). A concentration index is a mapping function \( C(p) \) such that \( C(p) \) increases with concentration and verifies \( C(p^-) \leq C(p) \leq C(p^+) \) with

\[
p^+ = \left\{ \exists i_0 : p_{i_0}^+ = 1, p_i^+ = 0 \text{ if } i \neq i_0 \right\} \text{ and } p^- = \left\{ \forall i : p_i^- = 1/n \right\}.
\]

For example

- The Herfindahl index

\[
H(p) = \sum_{i=1}^n p_i^2
\]

- The Gini index \( G(p) \) measures the distance between the Lorenz curve of \( p \) and the Lorenz curve of \( p^- \).

- The Shannon entropy is defined as follows\(^5\):

\[
I(p) = - \sum_{i=1}^n p_i \ln p_i
\]

\(^{5}\text{Note that the concentration index is the opposite of the Shannon entropy.}\)
Managing the Risk Concentration

Risk Parity Optimization

We would like to build a portfolio such that

$$\text{RC} (F_j) \simeq \text{RC} (F_k)$$

for \((j, k) \in \mathcal{J}\).

The optimization problem becomes

$$x^* = \arg \min C (p)$$

u.c. \[
\begin{align*}
1^T x &= 1 \\
x &\succeq 0
\end{align*}
\]

with \(p = \{\text{RC} (F_j), j \in \mathcal{J}\}\).
Managing the Risk Concentration
An Example (Slide 19)

The lowest risk concentrated portfolio
\((H ≡ G ≡ I)\)

Optimal solution \((y^*, \tilde{y}^*)\)

<table>
<thead>
<tr>
<th>(F_i)</th>
<th>RC((F_i))</th>
<th>RC*((F_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_1)</td>
<td>91.97% 7.28% 33.26%</td>
<td></td>
</tr>
<tr>
<td>(F_2)</td>
<td>25.78% 7.28% 33.26%</td>
<td></td>
</tr>
<tr>
<td>(F_3)</td>
<td>42.22% 7.28% 33.26%</td>
<td></td>
</tr>
</tbody>
</table>

- \(\tilde{F}_1\) 6.74% 0.05% 0.21%
- \(\tilde{\sigma}(\tilde{y})\) 23.41%

Corresponding portfolio \(x^*\)

<table>
<thead>
<tr>
<th>(A_i)</th>
<th>RC(_i)</th>
<th>RC(_i^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>0.30% 0.05% 0.22%</td>
<td></td>
</tr>
<tr>
<td>(A_2)</td>
<td>39.37% 9.11% 41.63%</td>
<td></td>
</tr>
<tr>
<td>(A_3)</td>
<td>0.31% 0.07% 0.30%</td>
<td></td>
</tr>
<tr>
<td>(A_4)</td>
<td>60.01% 12.66% 57.85%</td>
<td></td>
</tr>
</tbody>
</table>

- \(\tilde{\sigma}(x)\) 21.88%

With some constraints
\((H \neq G \neq I)\)

Optimal portfolios with \(x_i \geq 10\%\)

<table>
<thead>
<tr>
<th>Criterion</th>
<th>(H)</th>
<th>(G)</th>
<th>(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>10.00% 10.00% 10.00%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_2)</td>
<td>22.08% 18.24% 24.91%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_3)</td>
<td>10.00% 10.00% 10.00%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_4)</td>
<td>57.92% 61.76% 55.09%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- \(\tilde{H}^*\) 0.0436 0.0490 0.0453
- \(\tilde{\sigma}(\tilde{x})\) 23.41%
- \(\tilde{\sigma}(\tilde{y})\) 23.41%

- \(\bar{\sigma}(\bar{y})\) 23.41%
- \(\bar{\sigma}(\bar{x})\) 23.41%
Solving Invariance Problems of Choueifaty et al. (2011)

The Duplication Invariance Property

- $\Sigma^{(n)}$ is the covariance matrix of the $n$ assets.
- $x^{(n)}$ is the RB portfolio with risk budgets $b^{(n)}$.
- We suppose now that we duplicate the last asset

$$
\Sigma^{(n+1)} = \begin{pmatrix}
\Sigma^{(n)} & \Sigma^{(n)}e_n \\
e_n^T \Sigma^{(n)} & 1
\end{pmatrix}
$$

- We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = 0$ and $A = (I_n e_n)^T$.
- We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$.
- We have $x_i^{(n+1)} = x_i^{(n)}$ if $i < n$ and $x_n^{(n+1)} + x_{n+1}^{(n+1)} = x_n^{(n)}$.

$\Rightarrow$ The ERC portfolio verifies the duplication invariance property if the risk budgets are expressed with respect to factors and not to assets.
We introduce an asset $n + 1$ which is a positive linear (normalized) combination $\alpha$ of the first $n$ assets:

$$\Sigma^{(n+1)} = \begin{pmatrix} \Sigma^{(n)} & \Sigma^{(n)} \alpha \\ \alpha^\top \Sigma^{(n)} & \alpha^\top \Sigma^{(n)} \alpha \end{pmatrix}$$

We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = 0$ and $A = (I_n \quad \alpha)^\top$.

We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$.

We have $x_i^{(n)} = x_i^{(n+1)} + \alpha_i x_{n+1}^{(n+1)}$ if $i \leq n$.

$\Rightarrow$ RB portfolios (and so ERC portfolios) verify the polico invariance property if the risk budgets are expressed with respect to factors and not to assets.
The Fama-French Model

Framework

Capital Asset Pricing Model

\[ \mathbb{E}[R_i] = R_f + \beta_i (\mathbb{E}[R_{MKT}] - R_f) \]

where \( R_{MKT} \) is the return of the market portfolio and

\[ \beta_i = \frac{\text{cov}(R_i, R_{MKT})}{\text{var}(R_{MKT})} \]

Fama-French-Carhart model

\[ \mathbb{E}[R_i] = \beta_{i,MKT}^{MKT} \mathbb{E}[R_{MKT}] + \beta_{i,SMB}^{SMB} \mathbb{E}[R_{SMB}] + \beta_{i,HML}^{HML} \mathbb{E}[R_{HML}] + \beta_{i,MOM}^{MOM} \mathbb{E}[R_{MOM}] \]

where \( R_{SMB} \) is the return of small stocks minus the return of large stocks, \( R_{HML} \) is the return of stocks with high book-to-market values minus the return of stocks with low book-to-market values and \( R_{MOM} \) is the Carhart momentum factor.
The Fama-French Model

Regression Analysis

Results(*) using weekly returns from 1995-2012

<table>
<thead>
<tr>
<th>Index</th>
<th>$\beta_i^{MKT}$</th>
<th>$\beta_i^{SMB}$</th>
<th>$\beta_i^{HML}$</th>
<th>$\beta_i^{MOM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI USA Large Growth</td>
<td>1.06</td>
<td>-0.12</td>
<td>-0.38</td>
<td>-0.07</td>
</tr>
<tr>
<td>MSCI USA Large Value</td>
<td>0.97</td>
<td>-0.21</td>
<td>0.27</td>
<td>-0.12</td>
</tr>
<tr>
<td>MSCI USA Small Growth</td>
<td>1.04</td>
<td>0.64</td>
<td>-0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>MSCI USA Small Value</td>
<td>1.01</td>
<td>0.62</td>
<td>0.30</td>
<td>-0.10</td>
</tr>
</tbody>
</table>

(*) All estimates are significant at the 95% confidence level.

Question: What is exactly the meaning of these figures?
The Fama-French Model
Risk Contribution Analysis
The Fama-French Model
Risk Analysis of Long/Short Portfolios

(100%, -100%, 0%, 0%)

(0%, 0%, 100%, -100%)

(-100%, 0%, 100%, 0%)

(50%, 50%, -50%, -50%)
### The Risk Factors of the Yield Curve

**Principal Component Analysis**

**PCA factors**

1. Level
2. Slope
3. Convexity

#### US yield curve (2003-2012)

![Graph showing US yield curve with PCA factors](image)

#### Portfolio weights

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Maturity (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>#1</td>
<td>1</td>
</tr>
<tr>
<td>#2</td>
<td>-2</td>
</tr>
<tr>
<td>#3</td>
<td>10</td>
</tr>
<tr>
<td>#4</td>
<td>53</td>
</tr>
</tbody>
</table>
The Risk Factors of the Yield Curve

Risk Decomposition of the Four Portfolios wrt Zero-Coupons and PCA Risk Factors

Portfolio #1

Portfolio #2

Portfolio #3

Portfolio #4
Diversifying a Portfolio of Hedge Funds

Framework

▶ We consider the Dow Jones Credit Suisse AllHedge index\(^6\).
▶ We use three risk measures
  1. Volatility;
  2. Expected shortfall with a 80% confidence level;
  3. Cornish-Fisher value-at-risk with a 99% confidence level.
▶ Factors are based on PCA (Fung and Hsieh, 1997).
▶ We consider two risk parity models.
  1. ERC portfolio.
  2. Risk factor parity (RFP) portfolio by minimizing the risk concentration between the first 4 PCA factors.

\(^6\)This index is composed of 10 subindexes: (1) convertible arbitrage, (2) dedicated short bias, (3) emerging markets, (4) equity market neutral, (5) event driven, (6) fixed income arbitrage, (7) global macro, (8) long/short equity, (9) managed futures and (10) multi-strategy.
Diversifying a Portfolio of Hedge Funds

The ERC Approach

Risk decomposition in terms of factors

Simulated performance
Diversifying a Portfolio of Hedge Funds

The Risk Factor Parity (RFP) Approach

Risk decomposition in terms of factors

Simulated performance

T. Roncalli & G. Weisang (Lyxor & ClarkU)
Risk parity approach ≡ a promising way for strategic asset allocation (see e.g. Bruder and Roncalli, 2012)

**ATP Danish Pension Fund**

“Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk” (Henrik Gade Jepsen, CIO of ATP, IPE, June 2012).

These risk budgets are then transformed into asset classes’ weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities (Source: FTfm, June 10, 2012).
### Strategic Asset Allocation

**Risk Budgeting Policy of a Pension Fund**

#### Asset Allocation

<table>
<thead>
<tr>
<th>Asset class</th>
<th>RB</th>
<th>RB*</th>
<th>MVO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_i$</td>
<td>$RC_i$</td>
<td>$x_i$</td>
</tr>
<tr>
<td>US Bonds</td>
<td>36.8%</td>
<td>100%</td>
<td>45.9%</td>
</tr>
<tr>
<td>EURO Bonds</td>
<td>21.8%</td>
<td>10.0%</td>
<td>8.3%</td>
</tr>
<tr>
<td>IG Bonds</td>
<td>14.7%</td>
<td>15.0%</td>
<td>13.5%</td>
</tr>
<tr>
<td>US Equities</td>
<td>10.2%</td>
<td>20.0%</td>
<td>10.8%</td>
</tr>
<tr>
<td>Euro Equities</td>
<td>5.5%</td>
<td>10.0%</td>
<td>6.2%</td>
</tr>
<tr>
<td>EM Equities</td>
<td>7.0%</td>
<td>15.0%</td>
<td>11.0%</td>
</tr>
<tr>
<td>Commodities</td>
<td>3.9%</td>
<td>10.0%</td>
<td>4.3%</td>
</tr>
</tbody>
</table>

**RB* = A Black-Litterman portfolio with a tracking error of 1% wrt RB**

**MVO = Markowitz portfolio with the RB* volatility**
Combining the risk budgeting approach to define the asset allocation and the economic approach to define the factors (Kaya et al., 2011).

Following Eychenne et al. (2011), we consider 7 economic factors grouped into four categories:

1. activity: gdp & industrial production;
2. inflation: consumer prices & commodity prices;
3. interest rate: real interest rate & slope of the yield curve;
4. currency: real effective exchange rate.

Quarterly data from Datastream.

Risk budget matching (Sequential Quadratic Programming) using YoY relative variations for the study period Q1 1999 – Q2 2012.

Risk measure: volatility.
## Strategic Asset Allocation

### Allocation Between Asset Classes

- 13 AC: equity (US, EU, UK, JP), sovereign bonds (US, EU, UK, JP), corporate bonds (US, EU), High yield (US, EU) and US TIPS.

- Three given portfolios
  - Portfolio #1 is a balanced stock/bond asset mix.
  - Portfolio #2 is a defensive allocation with 20% invested in equities.
  - Portfolio #3 is an aggressive allocation with 80% invested in equities.

- Portfolio #4 is optimized in order to take more inflation risk.

<table>
<thead>
<tr>
<th>Factor</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity</td>
<td>36.91%</td>
<td>19.18%</td>
<td>51.20%</td>
<td>34.00%</td>
</tr>
<tr>
<td>Inflation</td>
<td>12.26%</td>
<td>4.98%</td>
<td>9.31%</td>
<td>20.00%</td>
</tr>
<tr>
<td>Interest rate</td>
<td>42.80%</td>
<td>58.66%</td>
<td>32.92%</td>
<td>40.00%</td>
</tr>
<tr>
<td>Currency</td>
<td>7.26%</td>
<td>13.04%</td>
<td>5.10%</td>
<td>5.00%</td>
</tr>
</tbody>
</table>

- Residual factors

<table>
<thead>
<tr>
<th>US</th>
<th>EU</th>
<th>UK</th>
<th>JP</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>20%</td>
<td>20%</td>
<td>5%</td>
</tr>
<tr>
<td>#2</td>
<td>10%</td>
<td>10%</td>
<td></td>
</tr>
<tr>
<td>#3</td>
<td>30%</td>
<td>30%</td>
<td>10%</td>
</tr>
<tr>
<td>#4</td>
<td>19.0%</td>
<td>21.7%</td>
<td>6.2%</td>
</tr>
</tbody>
</table>
Conclusion

- Risk factor contribution = a powerful tool.
- Risk budgeting with risk factors = be careful!
- PCA factors = some drawbacks (not always stable).

- Economic and risk factors = make more sense for long-term investment policy.
- Could be adapted to directional risk measure (e.g. expected shortfall).
- How to use this technology to hedge or be exposed to some economic risks?

- Our preliminary results open a door toward rethinking the long-term investment policy of pension funds.
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