Uncertain Covariance Models

RISK FORECASTS THAT KNOW HOW ACCURATE THEY ARE AND WHERE

AUG 18, 2015 QWAFAFEW BOSTON

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The Math of Uncertain Covariance

Straightforward ... and not for a presentation

Shah, A. (2015). Uncertain Covariance Models

http://ssrn.com/abstract=2616109

Questions or comments, please email AnishRS@InvestmentGradeModeling.com

$\begin{split} s^{\mu} &= (p \times 1) second start gravity or the second start gravity of the second start gravity start gravity of the second start gra$	where $\mathbf{v} = (z, x, T)$ perficitly samples $d\mathbf{z}_{0} = d^{2}\mathbf{z}_{0}$ $d_{0}(\mathbf{z}) = d^{2}\mathbf{z}_{0}$ $d_{0}(\mathbf{z}) = arc \left(d_{0}(\mathbf{z}_{0}, \mathbf{z}_{0}) \right)$ $= arc \left(\sum_{i} z_{i} \mathbf{z}_{i}, \sum_{i} w_{i} \mathbf{z}_{i}\right)$ $= a^{2} \mathbf{X}_{i} / (z = z)$ The same and manufacture material, the solubilities samplifies: $\mathbf{E}_{0}(\mathbf{z}) = \sum_{i} u \mathbf{E}_{i}$ 3. Either or the specifies Product The returns of the perfolder, interact free orthogonal materials free orthogonal systems.	$\begin{aligned} & = \frac{1}{2} r f(\mathbf{r}_{n}) r f(\mathbf{r}_{n}) \\ & = \sum_{n} r f(\mathbf{r}_{n}) r f(\mathbf{r}_{n}) \\ & = \sum_{n} \bar{\mathbf{r}}_{n} \mathbf{r}_{n} \end{aligned} \tag{6}$	$\begin{aligned} & \text{Representations of problem } \\ & = \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & \text{regions of } \frac{1}{p_n} = \sigma(p) \\ & \text{regions of } \frac{1}{p_n} \sum_{n} w_n^n \sigma(p) \\ & \text{regions of } \frac{1}{p_n} \sum_{n} w_n^n \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & = \sigma(p) \sum_{n} w_n^n \sigma(p) \\ & \text{regions of } \frac{1}{p_n} \sum_{n} w_n^n \sigma(p) $
$F = G d G^{*}$ -thread-composed $F^{+} = G d^{+} G^{*}$ $M = F^{+}$ $M^{*} M = F$	$f(\mathbf{w}) = \operatorname{retriever} of perturbative factor values = \sum_{k} \mathbf{e}_{k}^{k}$	$\begin{split} & \max\{f\} = g \max\{f_i(g)\} + \max\{g(g)\} \\ & = g \left[\sum_{i,k} g_i g_k g_{i,k} \right] + \min\left[\sum_{k} g_k g_{k} \right] \end{split}$	 p²0 + cur(p))0 + Connecting stade specific a broughter stades," lightening rig correlation between the sur-
$\label{eq:static_state} \begin{split} & M^{-M} = \beta \ & (2M^{-1})(2M^{-1})^{*} \\ & R_{2M} & = 2M^{-1} \\ & P_{2M} = \beta \\ & P_{2M} = \beta \ & \\ & P_{2M} = 1 \\ \end{split}$ Tradicting factorizes as responses to address to dauge of tradicts as one process to address to dauge of tradicts as one process to address to dauge of tradicts as the process of the state of the process of the pro	Variance (considing to the table), if postful the factor where $= \pi \left[\log \left[\exp(\exp(\log \log n) \right] \right] \right]^{-\frac{1}{2}}$ $= \pi \left[\sum_{k=1}^{n-1} \sigma_{k}^{k} \right]$	$\begin{split} &= \sum_{i,j} \sigma^i [a_i a_{ij}] c_{j,j} + \mu^* \cos(\theta) \mu \\ &= \sum_{i,j} \sum_{i} \lim_{t \to +\infty} (a_{ij}, a_{ij}) + \overline{a}_i \overline{a}_{ij}] c_{ij,j} + \mu^* \cos(\theta) \mu \\ &= \overline{a}^* C \overline{a} + \sum_{i} \sin(\theta) \overline{a}_{ij}] c_{ij,i} + \mu^* \cos(\theta) \mu (3) \end{split}$	$\begin{aligned} y^{\text{prov}}_{[I] + ij} &= \overline{x}^{[I]} c^{*} \overline{x}^{*} + \sum_{j \neq i} c_{ij} \\ y^{\text{prove}}_{[I] + ij} \\ &= y^{\text{prove}}_{I} \end{aligned}$
$\begin{split} g_{Last}(x_{uuv}) & \mapsto g(x \in M^{-1} v_{uuv}) M^{-1} \\ & \equiv Contential Postfold Value Contential Scientific Value Contential Sciences propagate to perfold to response propagate to perfold to response to the set of the science of t$	$= \sum_{k} \frac{\delta^{(2)} \mathbf{e}_{k} + \delta \mathbf{e}_{k,k}}{\delta \mathbf{e}^{(2)} \mathbf{e}^{(2)} \mathbf{e}^{(2)}}$ $= \sum_{k} \frac{\mathbf{e}_{k}^{(2)} + \mathbf{E}_{k,k}}{\delta \mathbf{e}^{(2)} \mathbf{e}^{(2)} \mathbf{e}^{(2)}}$ $= \sum_{k} \frac{\delta^{(2)} \mathbf{e}^{(2)} \mathbf{e}^{(2)} \mathbf{e}^{(2)}}{\delta \mathbf{e}^{(2)} \mathbf{e}^{(2)}}$ (6)	Stock specific real-to-metobalize to perform reasons, free squares (2), in 2, int al, Equard to derive of stock specific real, as a reference al factors, and a, as represent to these.	$\begin{split} \mathcal{C}' &= \begin{bmatrix} \mathcal{C} & \phi \\ \phi \end{bmatrix} \\ \mathcal{L}' &= \begin{bmatrix} \mathcal{C} & \phi \\ \phi \end{bmatrix} \end{split}$ The result followed here: below out which specific
A. Furthelity in Factor Exponents Total regionaries with result \overline{z} and convince generic \overline{z}	$\begin{split} & \tau \mathbf{a} \lim_{k \to 0} \mathbf{a}^{1^{n}} \\ & = \sum_{j \in I} 2 \min\{a_j, a_k\} + 4 \mathcal{I}[a_j] \mathcal{I}[a_k] \min\{a_j, a_k\} \\ & = \sum_{j \in I} 2 B_{j,k}^{1} + 4 \hat{a}_j \hat{a}_j \hat{b}_j \hat{a}_j (j) \end{split}$	Redunately is repeated to a repeate a billion Association on on a control association and the Association of $a = N[k,diag(a)]$ Say workship to be have reason a perfectly constant.	 finite scenarios of varia- max made as the shape of it manyle, the same of the solution is then by fisher a similar manylele same.
$\label{eq:result} \begin{array}{l} F = (p, x, f) \mbox{ security exponent integrations of F} \\ \overline{F} = (p, x, f) \mbox{ for each } f \in F$ \\ \overline{X}_{n,n}(h_i) = core(X_{n,i}, X_{n,j}) \\ = kreage core labors integrate in the terms of the form integrate integrate in the terms of the form integrate in the term integrate integrate in the term integrate integ$	18 set that $g_{\mu \mu}$. This fait $B_{\mu \mu}$ appear is separately in variant, spatian (4). The lass soft a possible for semantary, is more from severative graphs for $F^{(2)}_{\mu} = S^{(2)}_{\mu}$. This some find the selected possible of spatiag separate and spatiag is required; instability satisfy reflection of the scattering by requires	i.e. die wahe of statistispandie offsate energie met inter by the nerve moneth for all materiale. Refere energiese is tilten als in entities 23, hat perfond correlation collegene en oder inte over e(se, y) = restatist variation statist operfile effects	V torsail Non, for markening a sin pl matricely. Significant A and reasons, and separate 10, for Disconteness can be former matricely. Where its first on
For equations $\begin{split} & e(w) = (0, x, t) \text{ particular's equations} \\ & = V[w_1, 0] \text{ particular's equations} \\ & = V[w_2, 0] w_1(w_2) \\ \end{split} $	dana i darandy ang' Daga bas angani a in ina an ari ini dalamata ng ina anganj palika a na angan any anjah ang	 The spectra expectation of give forwards beaming to an end of the spectra expectation and give forwards for an end of the spectra of the spectra of the spectra of the spectra of the spectra of the spectra of the spectra of the spectra of the spectra of the spectra of the spectr	 Partie and Specific Part 2 T the models are estimated particle displacian paths indexis and late the accuracy

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Background: Why Care About Covariance?

Notions of co-movement are needed to make decisions throughout the investment process

- 1. Estimating capital at risk and portfolio volatility
- 2. Hedging
- 3. Constructing and rebalancing portfolios through optimization
- 4. Algorithmic trading
- 5. Evaluating performance
- 6. Making sense of asset allocation

Background: Why Care About Uncertainty?

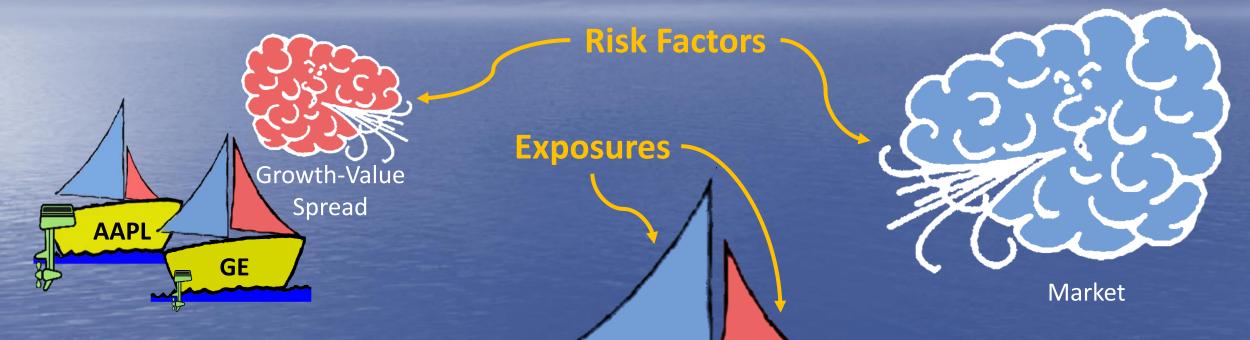
- 1. Nothing is exactly known. Everything is a forecast
- 2. However, one can estimate the accuracy of individual numbers Suppose two stocks have the same expected covariance against other securities The first has been well-predicted in the past, the other not Which is a safer hedge?
- It's imprudent to make decisions without considering accuracy
 A fool, omitting accuracy from his objective, curses optimizers for taking numbers at face value

4. Relying on wrong numbers can cost you your shirt

under leverage

when you can be fired and have assets assigned to another manager

A Factor Covariance Model in Pictures



Together these constitute a model of how securities move – jointly (winds and sails) and independently (motors)

GOOG

Stock-Specific

Effect

The Math of a Factor Covariance Model

- 1. Say a stock's return is partly a function of pervasive factors, e.g. the return of the market and oil $r_{GOOG} = h_{GOOG}(f_{mkt}, f_{oil}) + \text{stuff}$ assumed to be independent of the factors and other securities
- 2. Imagine linearly approximating this function

$$r_{GOOG} \approx \frac{\partial h_{GOOG}}{\partial f_{mkt}} f_{mkt} + \frac{\partial h_{GOOG}}{\partial f_{oil}} f_{oil} + \text{constant} + \text{stuff}$$

3. Model a stock's variance as the variance of the approximation

$$var(r_{GOOG}) \approx \underbrace{\begin{bmatrix} \frac{\partial h_{GOOG}}{\partial f_{mkt}} & \frac{\partial h_{GOOG}}{\partial f_{oil}} \end{bmatrix}}_{exposures} \underbrace{\begin{bmatrix} var(f_{mkt}) & cov(f_{mkt}, f_{oil}) \\ cov(f_{mkt}, f_{oil}) & var(f_{oil}) \end{bmatrix}}_{factor covariance} \underbrace{\begin{bmatrix} \frac{\partial h_{GOOG}}{\partial f_{mkt}} \\ \frac{\partial h_{GOOG}}{\partial f_{oil}} \end{bmatrix}}_{exposures} + \underbrace{\underbrace{var(stuff)}_{stock specific variance} \\ size of motor \\ size of motor \\ sails}$$

rah

The Math of a Factor Covariance Model

4. Covariance between stocks is modeled as the covariance of their approximations

$$cov(r_{GOOG}, r_{GE}) \approx \underbrace{\begin{bmatrix} \frac{\partial h_{GOOG}}{\partial f_{mkt}} & \frac{\partial h_{GOOG}}{\partial f_{oil}} \\ exposures \\ sails (GOOG) \end{bmatrix}}_{exposures} \underbrace{\begin{bmatrix} var(f_{mkt}) & cov(f_{mkt}, f_{oil}) \\ cov(f_{mkt}, f_{oil}) & var(f_{oil}) \end{bmatrix}}_{factor covariance \\ how wind blows} \underbrace{\begin{bmatrix} \frac{\partial h_{GOOG}}{\partial f_{mkt}} \\ \frac{\partial h_{GE}}{\partial f_{oil}} \end{bmatrix}}_{exposures \\ sails (GE) \underbrace{\begin{bmatrix} \frac{\partial h_{GOOG}}{\partial f_{mkt}} \\ \frac{\partial h_{GE}}{\partial f_{oil}} \end{bmatrix}}_{exposures}$$

r ah = r

Note: no motors here – they are assumed independent across stocks

- 5. Parts aren't known but inferred, typically by regression, or by more sophisticated tools
- 6. Best is to forecast values over one's horizon

An Uncertain Factor Covariance Model

Welcome to reality! Nothing is known with certainty. But some forecasts are believed more accurate than others

A portfolio's risk has – according to beliefs – an expected value and a variance

Good decisions come from considering uncertainty explicitly. Ignoring doesn't make it go away



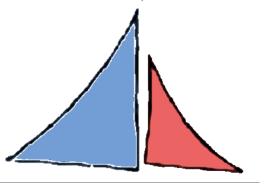
Uncertain Exposures



Uncertain Risk Factors projected onto orthogonal directions



Uncertain Stock-Specific Effect



Uncertain Exposures

Beliefs about future exposure to the factors are communicated as Gaussian

- $\boldsymbol{e}_{GOOG} \sim N[\hat{\boldsymbol{e}}_{GOOG}, \widehat{\boldsymbol{\Omega}}_{GOOG}]$
- Exposures can be correlated across securities
- Estimates of mean and covariance come from the method to forecast exposures and historical accuracy

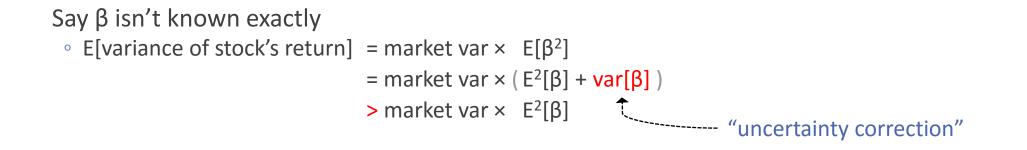
So, a portfolio's exposures are also Gaussian

• This fact is used in the math to work out variance (from uncertainty) of portfolio return variance

Sidebar: E[β²] > (E[β])² Ignoring Uncertainty Underestimates Risk

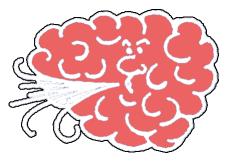
A CAPM flavored bare bones example to illustrate the idea:

- Stock's return = market return $\times \beta$
- $\,\circ\,$ Variance of stock's return = market var $\times\,\beta^2$



Ignoring uncertainty underestimates risk

• Note: this has nothing to do with aversion to uncertainty



Uncertain Factor Variances

Beliefs about the future factor variances are communicated as their mean and covariance

- Forecasts are the mean and covariance according to uncertainty of return variances
- Not Gaussian since variances ≥ 0

How the heck do you generate these?

Shah, A. (2014). Short-Term Risk and Adapting Covariance Models to Current Market Conditions

<u>http://ssrn.com/abstract=2501071</u>

- 1. Forecast whatever you can, e.g. from VIX and cross-sectional returns, the volatility of S&P 500 daily returns over the next 3 months will be 25% ± 5% annualized
- 2. The states of quantities measured by the risk model imply a configuration of factor variances

Since this inferred distribution of factor variances arises from predictions, it is a forecast

Adapting Infers Distribution Of Factor Variances

SP 500 IBM

1. Noisy variance forecasts via all manner of Information sources

Implied vol, intraday price movement, news and other big data, ...

2. Imply a distribution on how the world is

FD)

3. An aside: this extends to the behavior of other securities All risk forecasts are improved

SB UX

Ο



Stock-specific effects (motors) are best regarded as both exposures (sails) and factors (wind)

- 1. A motor is like wind that affects just 1 stock
- 2. Its size is estimated with error like a sail
- The average size of motors across securities varies over time

 e.g. stock-specific effects can shrink as market volatility rises
 Since it might depend on the variance of other factors, average size gets treated like one

Thus, uncertainty from stock-specific effects is captured using both types of uncertainty in the preceding slides.

All Set with Machinery: Uncertain Portfolio Variance

Math then yields for a portfolio

- expected variance
- standard deviation of variance

according to beliefs (estimates) about uncertainty in the pieces

- exposures
- factor variances
- stock-specific effects

Expectation and standard deviation are with respect to beliefs

- Not reality, but one's best assessment of it
- "Given my beliefs, the portfolio's tracking variance is E ± sd"
- What a person (or computer) needs to make good decisions from the information at hand

Applications

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Portfolio Optimization: Uncertain Utility

Conventional

```
\max_{\mathbf{w}} U(\mathbf{w}) = r(\mathbf{w}) - \lambda \times v(\mathbf{w}) where r(\mathbf{w}) = \text{mean return}, v(\mathbf{w}) = \text{variance of return}
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Uncertain

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\max_{\mathbf{w}} O(\mathbf{w}) = E[U(\mathbf{w})] - \gamma \times stdev[U(\mathbf{w})]var[U(\mathbf{w})] = var[r(\mathbf{w})] + \lambda^{2} var[v(\mathbf{w})] - 2 \lambda stdev[r(\mathbf{w})] stdev[v(\mathbf{w})] \times \rho_{\mathbf{w}}
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 $ρ_w = cor[r(w), v(w)]$ for portfolio w, the correlation of uncertainty in mean and in variance $r(w) \sim N[w^T \mu, w^T \Sigma w]$ assume mean returns have Gaussian error

All pieces are known except ρ_w which, absent beliefs, can be set to 0

Portfolio Optimization: Maximize Risk Adjusted Return

Risk-adjusted return ≡ portfolio alpha - ½ portfolio tracking variance

Say alpha is known exactly

Randomly pick 10 securities – ½ are eq wt benchmark, ½ are optimized into fully invested portfolio

Optimize under the following covariance models

Base conventional Bayesian 15 factor PCA model

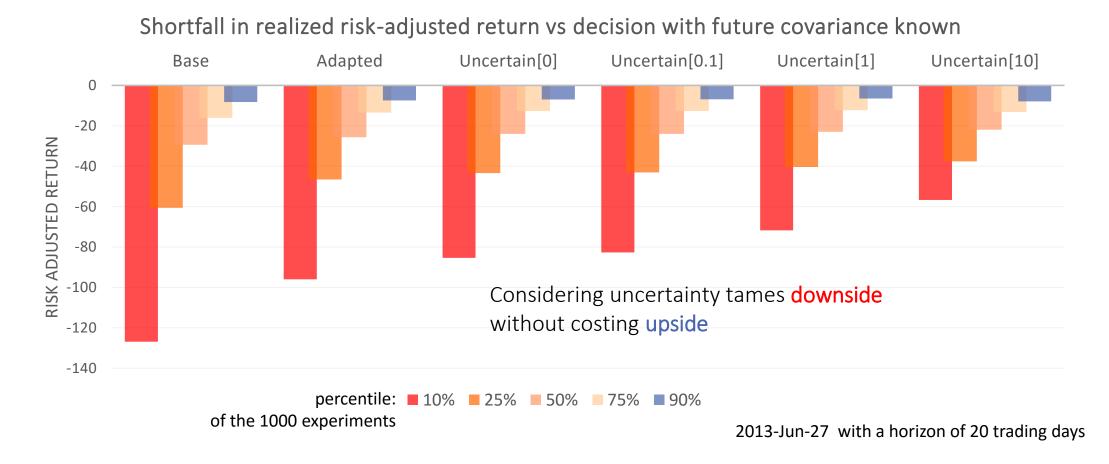
Adapted base model adapted to forecasts of future market conditions

Uncertain[y] adapted model with uncertainty information
maximize alpha - ½ [variance + y stdev(variance)]
note: Uncertain[0] has uncertainty correction, but no penalty on uncertainty

Measure the shortfall in realized risk-adjusted return vs. the ideal (w/future covariance known)

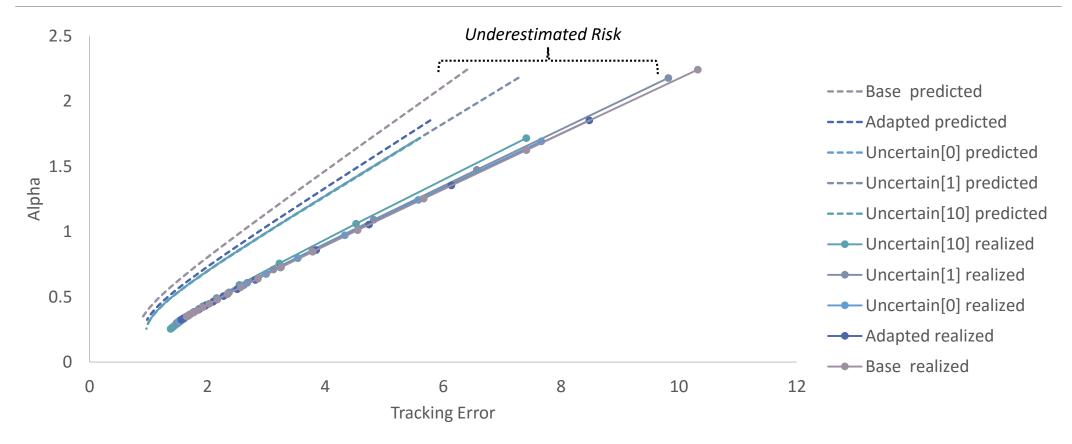
Repeat the experiment 1000 times

Portfolio Optimization: Maximize Risk Adjusted Return (cont)

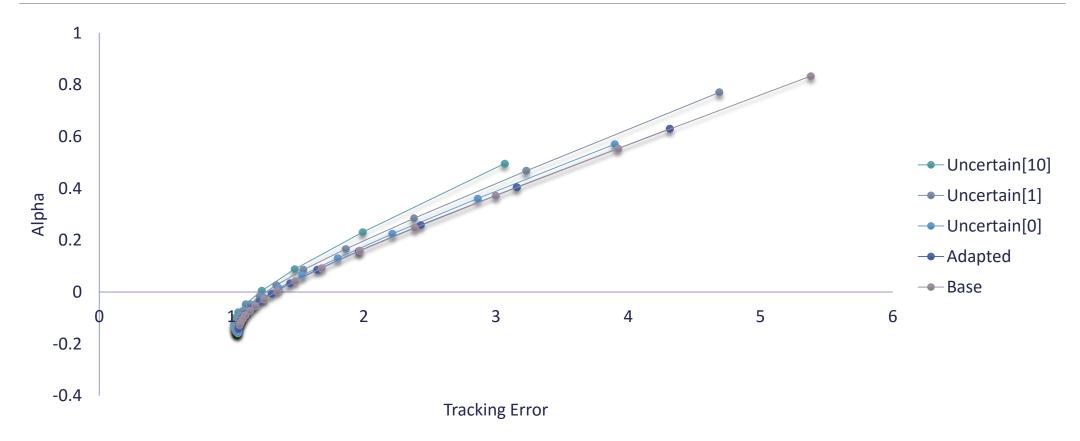


Predicted vs Realized Efficient Frontier

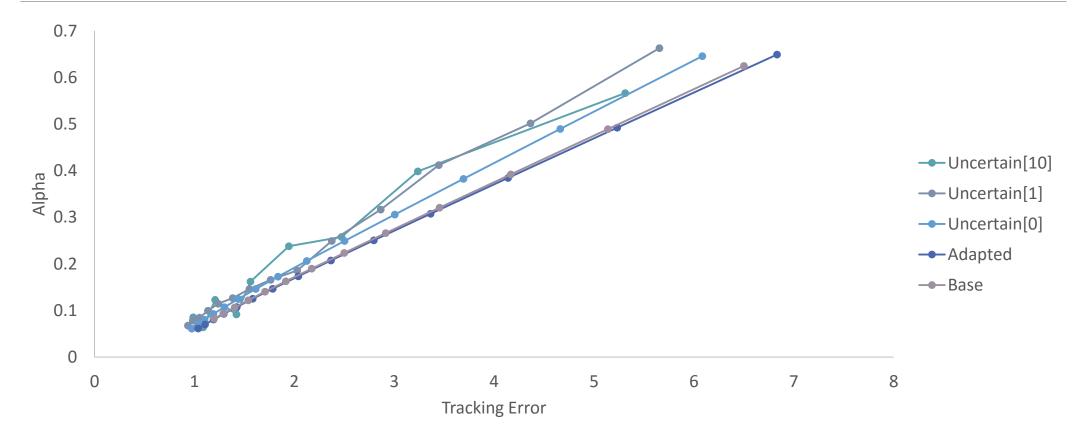
Perfectly known alphas Random 5 stock portfolio, 5 stock benchmark



Realized Efficient Frontier, Imperfect Alphas cor(Forecast, Realized Alpha) = .07 Random 5 stock portfolio, 5 stock benchmark



Realized Efficient Frontier, Imperfect Alphas cor(Forecast, Realized Alpha) = .07 Random 15 stock portfolio, 15 stock benchmark



Pairs Trading

Improved via knowledge of uncertainty

- Feel bullish (or bearish) about one or several similar securities
- Have candidates you feel the opposite about
- Choose from the two sets the long-short pair with lowest uncertainty penalized risk
- If you have explicit alpha forecasts, instead maximize uncertain utility
- Better risk control = safer leverage and more room to pursue alpha

A toy example, hedging with 3 securities

- AAPL is the reference security
- Every 11 trading days from 2012 through 2013, find the best 3 hedges (ignoring stock-specific effects) from a universe of tech stocks and equal weight-them
- Calculate the subsequent 10 day volatility of daily returns of long AAPL, short the equal weighted

Model	Avg 1 Day TE
Base	1.34
Uncertain[0]	1.13
Uncertain[0.5]	1.14
Uncertain[1]	1.13
Uncertain[100]	1.15

Investment Grade Modeling LLC Uncertain Covariance Models

Run nightly in the cloud. Bloomberg BBGID or ticker

Built uniquely for each use

- Risk factors arise from one's universe (e.g. only healthcare + tech, all US equities + commodity indices)
- .. and horizon (e.g. 1 day, 6 months) and return frequency (daily or weekly)
- Exposures (sails) are forecasts of the average over the horizon
- Factors volatilities (winds) and specific risks (motors) are adapted to forecasts of future volatility over the horizon made from broad set of information
- Though I believe risk crowding is baloney: zero chance of crowding from others using an identical model
- "Augmented" PCA
- PCA + (as necessary) factors to cover important not-stock-return-pervasive effects, e.g. VIX and certain commodities

Java library does uncertainty calculations

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