Boston QWAFAFEW

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Investment Horizon and Portfolio Selection

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Main results

- An objective function that addresses

__What__ do you need and __when__ do you need it?

- Horizon matters, i.e. your portfolio depends explicitly on investment horizon
  - *If you care about shortfall more than you care about surplus*
  - *Even if returns are iid*
- Tactical becomes strategic

Working paper

Investment Horizon and Portfolio Selection

Martin Tarlie, 2016

https://ssrn.com/abstract=2854336

Another working paper

Optimal Holdings of Active, Passive, and Smart Beta Strategies

Edmund Bellord, Joshua Livnat, Dan Porter, and Martin Tarlie

2017

https://ssrn.com/abstract=2987924

Agenda

1. Basic idea
2. Operationalizing the *what*
3. Operationalizing the *when*
4. Asset allocation example
5. Two period binomial model example
6. Risk aversion

What do you need and when do you need it?

Investment horizon

Why does investment horizon matter for your portfolio?

- Natural question

When do you need your money?

- An eternal asset allocation question

Are stocks more attractive in the long run?

Horizon sensitivity – conventional paradigm

When does your portfolio depend on horizon?

	Logarithmic Utility	Constant Relative Risk Aversion
iid returns	No	No
Mean reverting expected returns	No	**YES**

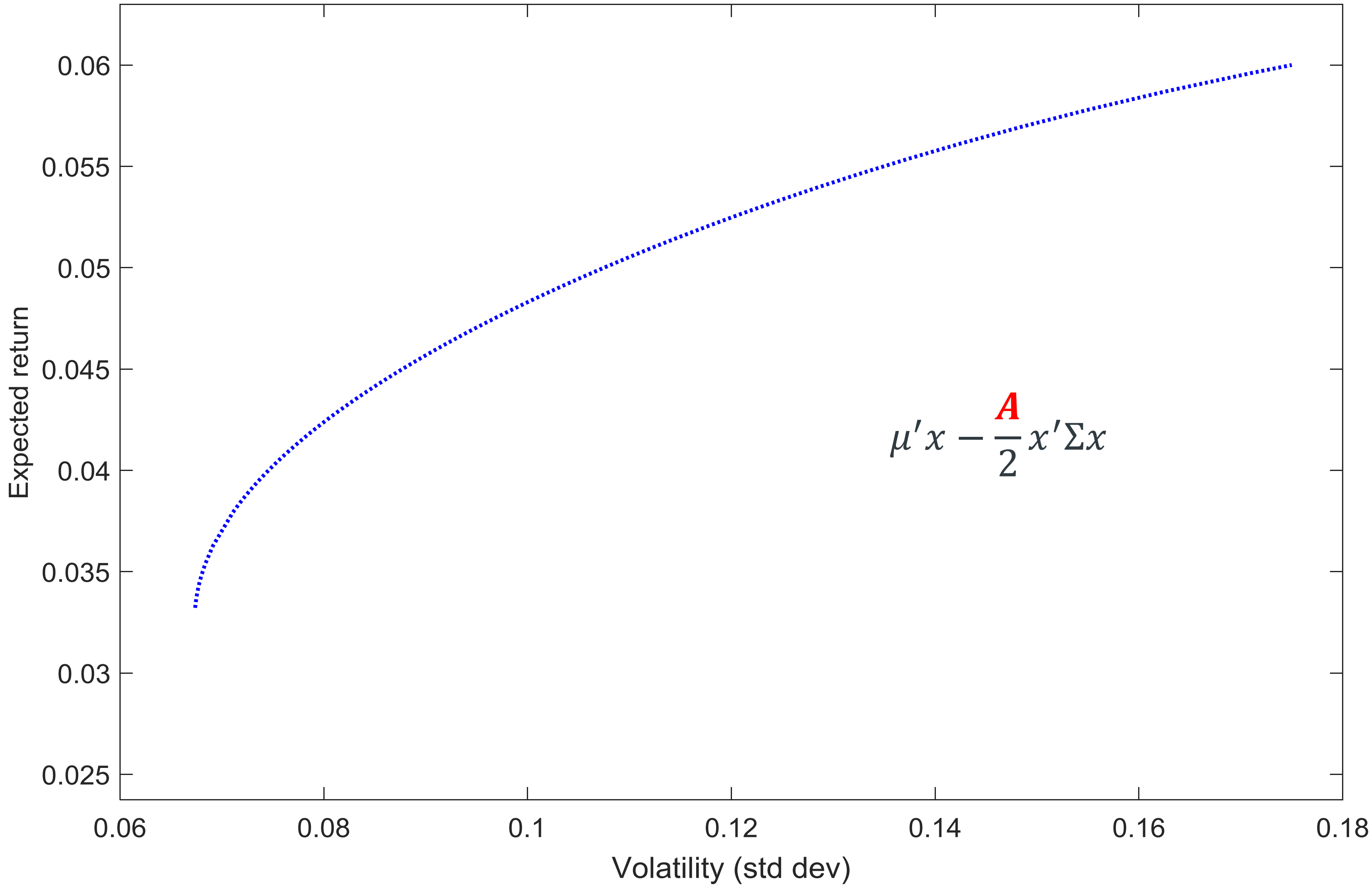
Horizon sensitivity

When does your portfolio depend on horizon?

	Logarithmic Utility	Constant Relative Risk Aversion	Piecewise Power Utility + Asymmetric Preferences
iid returns	No	No	YES
Mean reverting expected returns	No	YES	YES

Where on the efficient frontier?

How much do you care about the variability of your portfolio (or wealth)?



$$\mu' x - \frac{A}{2} x' \Sigma x$$

Extending mean variance

Investment risk is not having what you need when you need it

- Focusing on the *needs* and *circumstances* of the investor leads to an expanded set of questions

1. *What* do you need/desire?
2. How much do you care about not achieving your need/desire?
3. *What* do you have?
4. *When* do you need/desire it?

What do you need and when do you need it?

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What do you need and when do you need it?

Extending mean variance

Investment risk is not having what you need when you need it

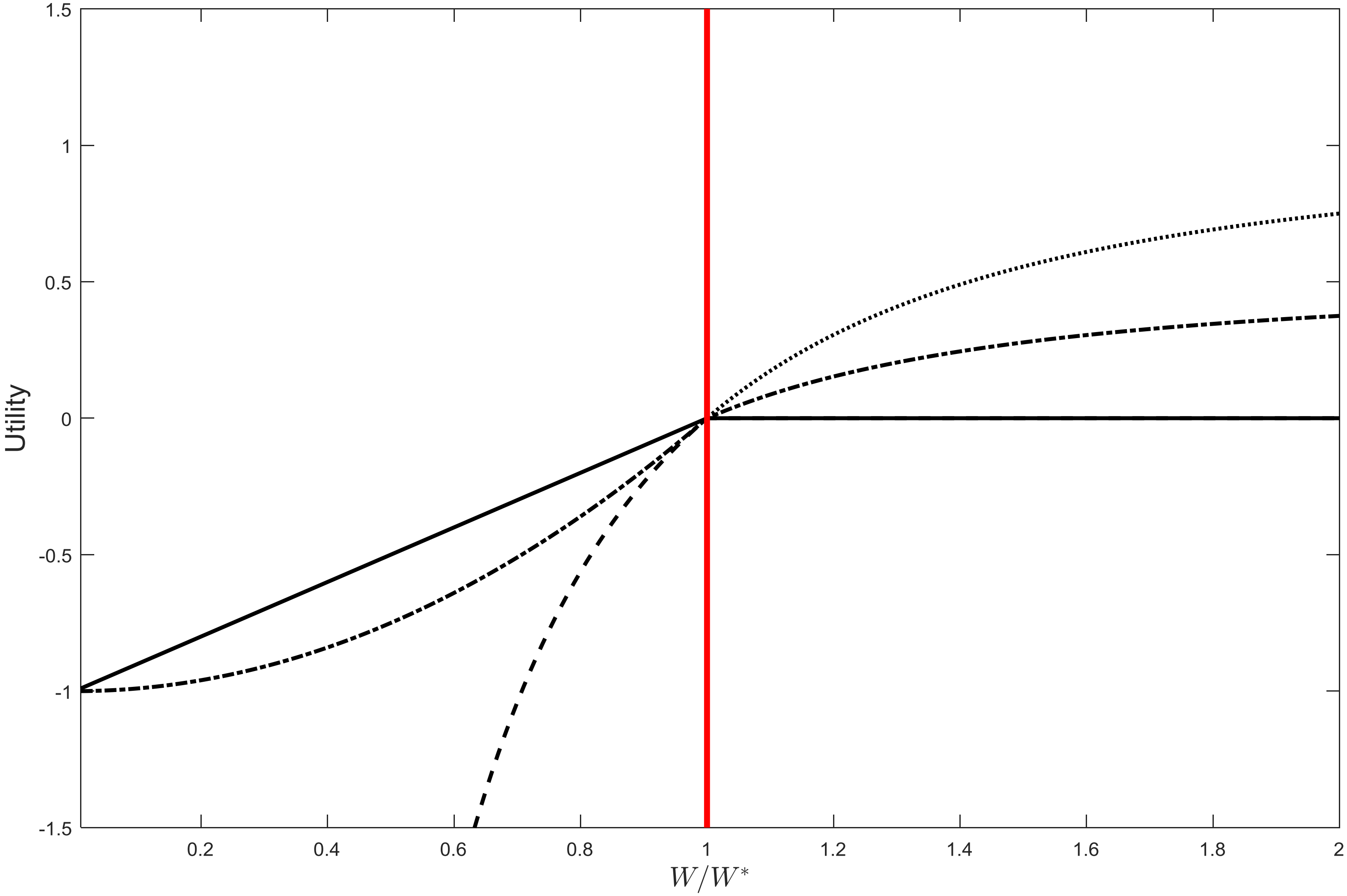
- Focusing on the *needs* and *circumstances* of the investor leads to an expanded set of questions

1. ***What* do you need/desire?**
   - **Introduce a wealth target**
2. **How much do you care about not achieving your need/desire?**
   - **Asymmetric preferences to shortfall and surplus**
3. *What* do you have?
4. *When* do you need/desire it?

What do you need and when do you need it?

Operationalizing the *what*

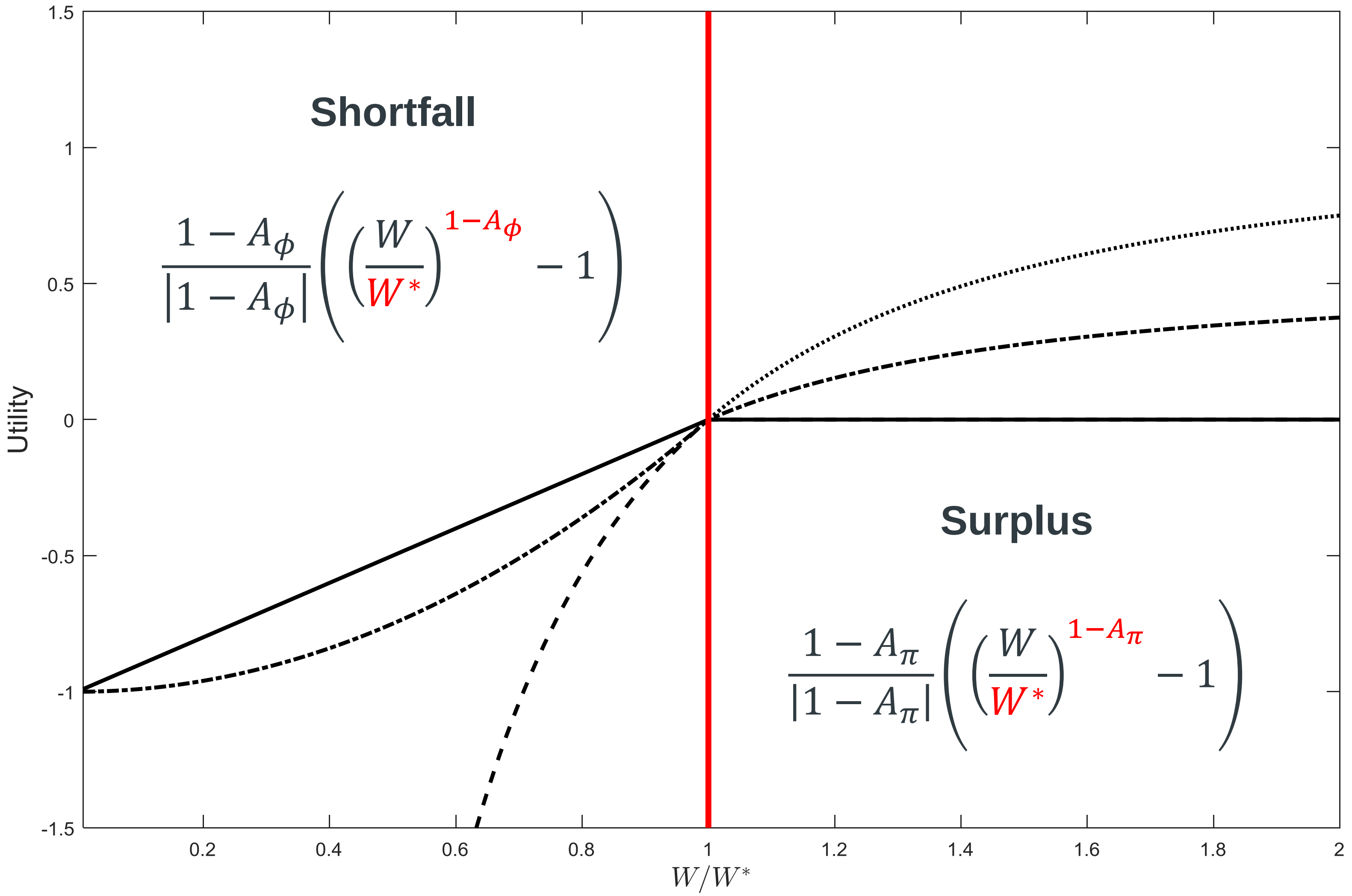
Wealth target + piecewise power law utility



W^* = **wealth target**

Operationalizing the *what* – the algebra

Wealth target + piecewise power law utility



Shortfall

$$\frac{1-A_\phi}{|1-A_\phi|}\left(\left(\frac{W}{W^*}\right)^{1-A_\phi}-1\right)$$

Surplus

$$\frac{1-A_\pi}{|1-A_\pi|}\left(\left(\frac{W}{W^*}\right)^{1-A_\pi}-1\right)$$

- Asymmetry between attitude to shortfall and surplus is the key driver of horizon sensitivity
- Risk aversion at the target is a measure of this asymmetry

Basic building block

Power law utility

$$U(W) = \frac{1-A}{|1-A|}\, W^{1-A}$$

- Workhorse utility in financial economics
- Risk aversion (Arrow-Pratt, relative)

$$A = -\frac{1}{W}\frac{U''}{U'}$$

  - Risk averse: $A > 0$
  - Risk seeking: $A < 0$
  - Risk neutral: $A = 0$
  - Note: W = wealth, , $A \neq 1, U' = \partial U / \partial W$

Power law utility and mean variance

Mean variance ~ power law utility

- Mean variance objective

$$\mu' x - \frac{A}{2} x' \Sigma x$$

- For *power law utility* and lognormally distributed wealth

$$E[W^{1-A}] = e^{(1-A)\left(\ln E[W] - \frac{A}{2} Var(\ln W)\right)}$$

This expression follows from

$$E[W^{1-A}] = e^{(1-A)E[\ln W] + \frac{(1-A)^2}{2} Var(\ln W)}$$

and using the fact that

$$\ln E[W] = E[\ln W] + \frac{Var(\ln W)}{2}$$

Expected shortfall

Focus on expected shortfall, analogous results for expected surplus

- Expected shortfall utility

$$\Phi = \int_0^{W^*} \left\{ \underbrace{\frac{1-A_\phi}{\left|1-A_\phi\right|}\left(\left(\frac{W}{W^*}\right)^{1-A_\phi} - 1\right)}_{\text{Utility of shortfall}} \right\} \underbrace{P(W)dW}_{\text{Probability of shortfall}}$$

  - The probability weighted sum of the utility of shortfall for all values of wealth less than the target
  - Wealth is lognormally distributed, so

$$P(W) = \frac{e^{-\frac{(\ln W - E[\ln W])^2}{2Var(\ln W)}}}{W\sqrt{2\pi Var(\ln W)}}$$

Expected utility – shortfall plus surplus

- Explicit objective function (see Appendix and working paper for details)

$$\alpha_\phi \overbrace{\Phi\left(\underbrace{\underbrace{E[\ln W]}_{\text{mean}},\underbrace{Var(\ln W)}_{\text{variance}}}_{\text{Depend on how you invest}};\underbrace{W^*, A_\phi}_{\text{Investor preferences}}\right)}^{\text{Shortfall}} + \alpha_\pi \overbrace{\Pi\left(E[\ln W],Var(\ln W);W^*,A_\pi\right)}^{\text{Surplus}}$$

  - Five preference parameters
    - W^* = target wealth
    - A_ϕ risk aversion below the target
    - A_π risk aversion above the target
    - α_ϕ and α_π are the weights on expected utility of shortfall and surplus

- Same ingredients as mean-variance objective function
  - Mean variance has a single risk aversion parameter

Interpretation of shortfall

- Expected shortfall accounts for probability *and* magnitude of shortfall

$$\Phi \;\sim -\underbrace{P(W < W^*)}_{\text{Probability of shortfall}} + \underbrace{E\left[\left(\frac{W}{W^*}\right)^{1-A_\phi}\right] P\left(W < W^* e^{-(1-A_\phi)Var(\ln W)}\right)}_{\text{Captures magnitude of shortfall}}$$

- This objective function nests the pure probability of shortfall objective

$$\lim_{A_\phi \to -\infty} \Phi \;\sim -P(W < W^*)$$

A general framework

Piecewise power law + asymmetric preferences

- An objective function that extends the mean-variance paradigm
- Applies to both absolute and relative problems
- Think balance sheet: wealth = assets – liabilities
- Liabilities define the benchmark
  - Consumption in retirement (think target date fund)
  - Plan sponsor with a policy benchmark
  - Pension plan with cash flow liabilities
  - S&P 500 for a benchmarked equity manager
- Natural extensions
  - Multiple wealth targets
  - Range of horizons, rather than a single point in time

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What do you need and when do you need it?

Operationalizing the *when*

To operationalize the when, need a model of asset returns

- (Log) asset returns are normally distributed – tied to our assumption of lognormally distributed wealth

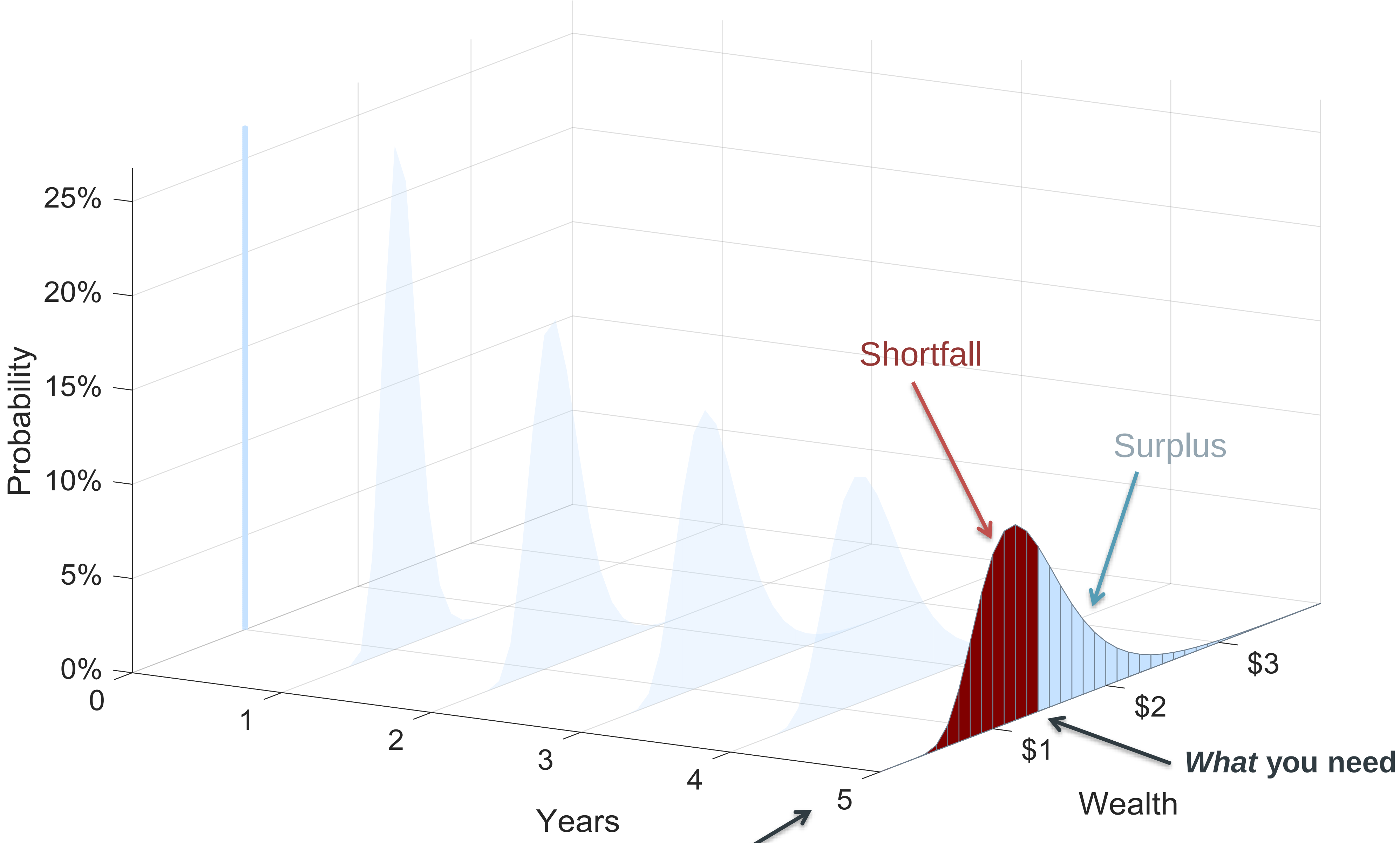
$$d\ ln\ R_i(t) = \underbrace{r_i(t)\ dt}_{\text{Expected return}} + \underbrace{\sigma_{Ri} dB_i^R(t)}_{\text{Unexpected return}}$$

- Wealth dynamics

$$W(t+dt) = \sum_i \underbrace{\{W(t)x_i(t)\}}_{\text{Amount invested in asset } i}\ \underbrace{e^{d\ \ln R_i(t)}}_{\text{Return on asset } i}$$

$dB_i^R(t)$ is a standard Wiener increment, and $\sum_i x_i(t) = 1$

Visualizing the *when*



***What* you need**

***When* you need it**

Target compounding rate

Once we introduce time, the target compounding rate emerges

- Target compounding rate (TCR)

$$\underbrace{W^*(T)}_{\textit{What}\text{ you need}} = \underbrace{W(0)}_{\textit{What}\text{ you have}} e^{\mathbf{TCR} * \underset{\nwarrow}{T}}$$

When you need it

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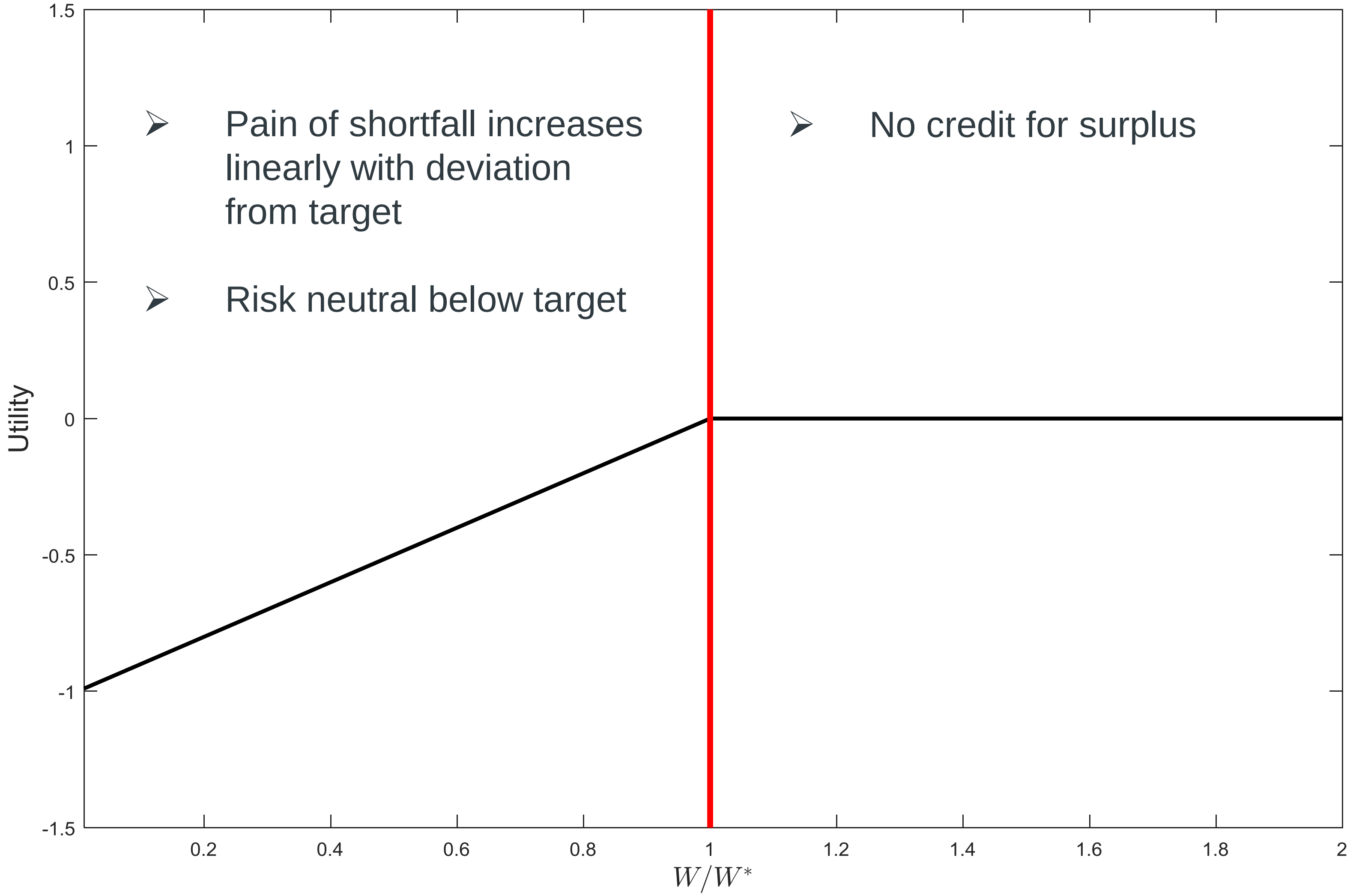
What do you need and when do you need it?

Utility function – pure shortfall

$$\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0$$



- Pain of shortfall increases linearly with deviation from target
- Risk neutral below target
- No credit for surplus

Asset returns – iid

- Constant expected returns

$$d\ ln\ R_i(t) = \underbrace{\bar{r}_i\ dt}_{\text{Constant expected return}} + \sigma_{Ri} dB_i^R(t)$$

- Stock and bond characteristics

	Stocks	Bonds
Expected real return ($\bar{r}_i$)	6%	2.5%
Volatility (σ_{Ri})	17.5%	7.3%

  – Assume return correlation equals zero

Objective function

Two ingredients – mean and variance – depend on investment choices

- Objective function

$$\Phi\left(\underbrace{\overbrace{E[\ln\ W\ (\boldsymbol{T})]}^{\text{Expected log wealth}},\overbrace{Var(\ln\ W\,(\boldsymbol{T}))}^{\text{Variance of log wealth}}}_{\text{Depend on how you invest}}\;;\underbrace{\mathrm{TCR},\ A_{\phi}}_{\textbf{Investor preferences}}\right)$$

  - Mean and variance depend on the weights $x_i(t)$ invested in asset i at time t
  - Explicit formulas for mean and variance for iid returns

$$Var(\ln W(\boldsymbol{T})) = \int_0^{\boldsymbol{T}} dt\, x'(t)\,\Sigma\, x(t)$$

$$E[\ln W(\boldsymbol{T})] = \ln W(0) + \int_0^{\boldsymbol{T}} dt\,\mu'\,x(t) - \frac{1}{2} Var(\ln W(\boldsymbol{T}))$$

Note: $\mu_i = \bar{r}_i + \frac{1}{2}\sigma_i^2$ is the expected arithmetic return on asset i

Optimization

Two ingredients – mean and variance – depend on investment choices

- Objective function Φ depends on the term structure of portfolios (i.e. $x_i(t)$) over the entire investment horizon

- ***Directly*** optimizing Φ generates a term structure of optimal portfolios $x_i^*(t)$ over the entire investment horizon

  - The portfolio that matters today is the “current” (i.e. $t = 0$) portfolio $x^*(0)$

  - The optimal portfolios for $t > 0$ reflect the stochastic evolution of wealth (c.f. two period binomial model)

An example – stock-bond allocation

Optimize pure shortfall utility function: $\boldsymbol{\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0}$

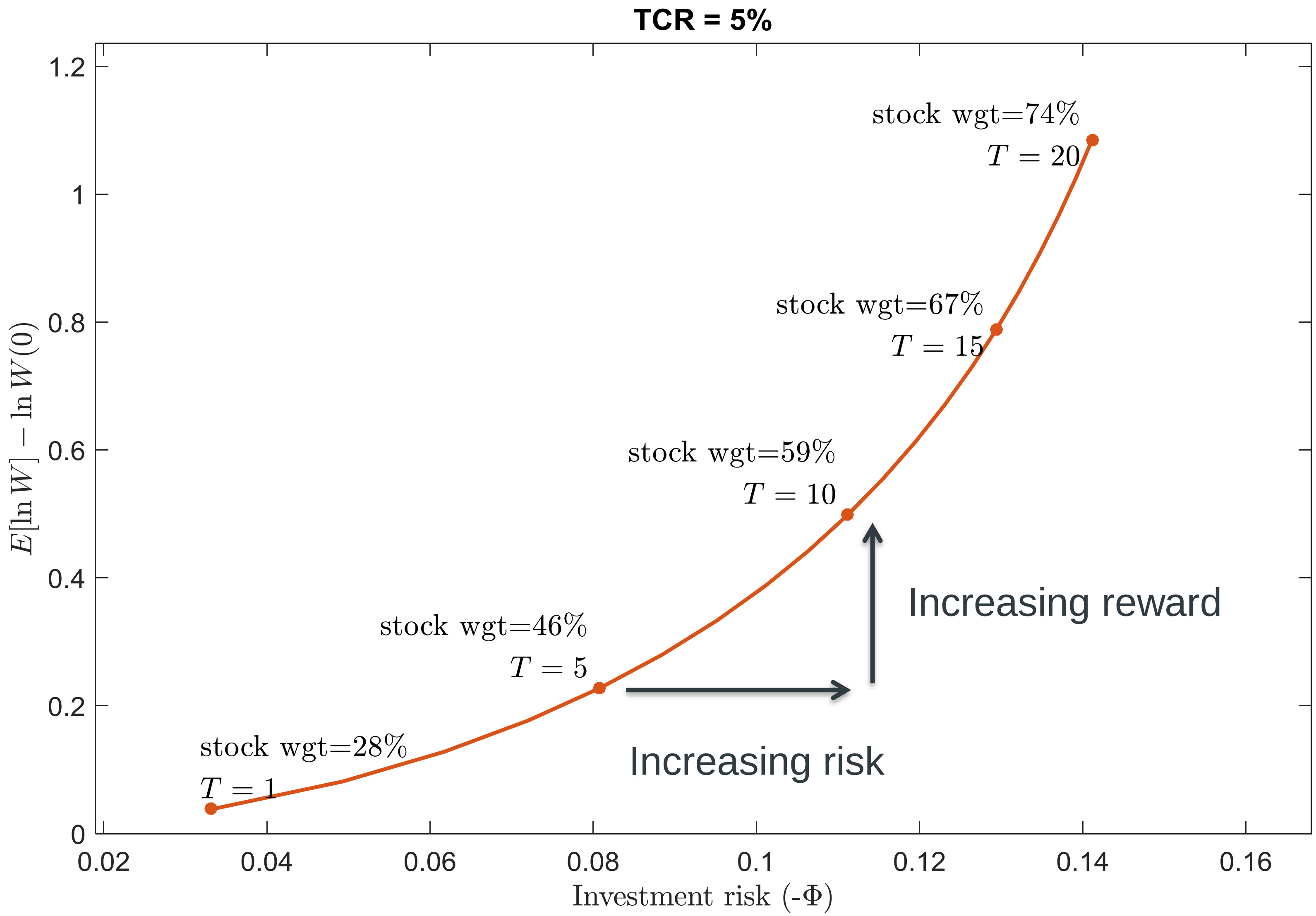
Stock weights $x^*(0)$ as a function of target compounding rate and horizon

Target Compounding Rate	T=1 yr	T=3	T=5	T=7	T=10	T=20
2%	24%	29%	31%	33%	35%	38%
3%	25%	31%	35%	38%	41%	47%
4%	26%	34%	40%	44%	49%	59%
5%	28%	38%	46%	52%	59%	74%
6%	29%	43%	53%	61%	70%	89%
7%	31%	49%	61%	71%	82%	104%

Assume return correlation of zero between stocks and bonds

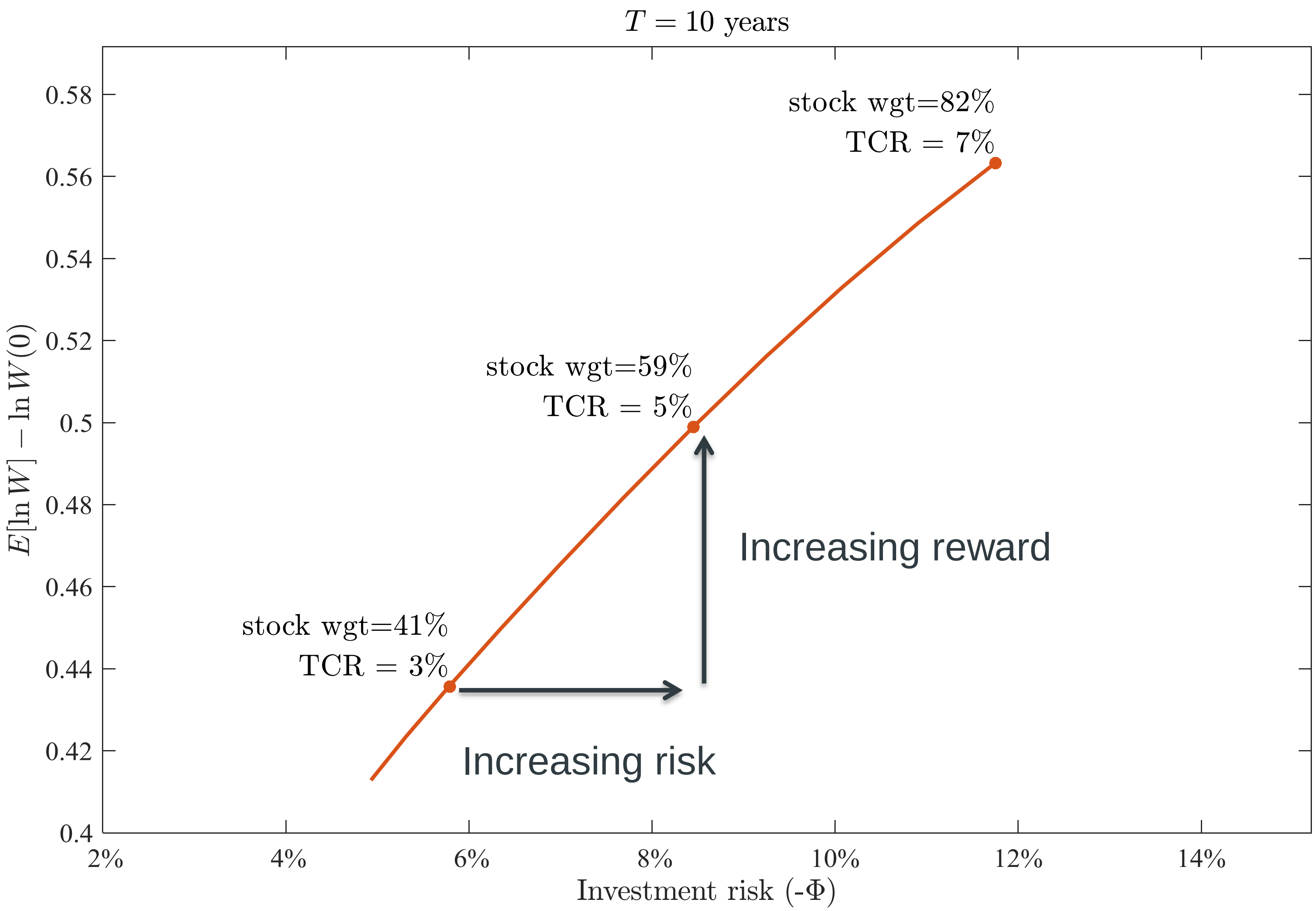
Reward and *investment* risk

Constant target compounding rate, vary horizon



Reward and *investment* risk

Constant horizon, vary target compounding rate (TCR)



Compare to pure power law (aka mean variance)

Optimize pure shortfall utility function: $\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0$

Stock weights as a function of target compounding rate and horizon

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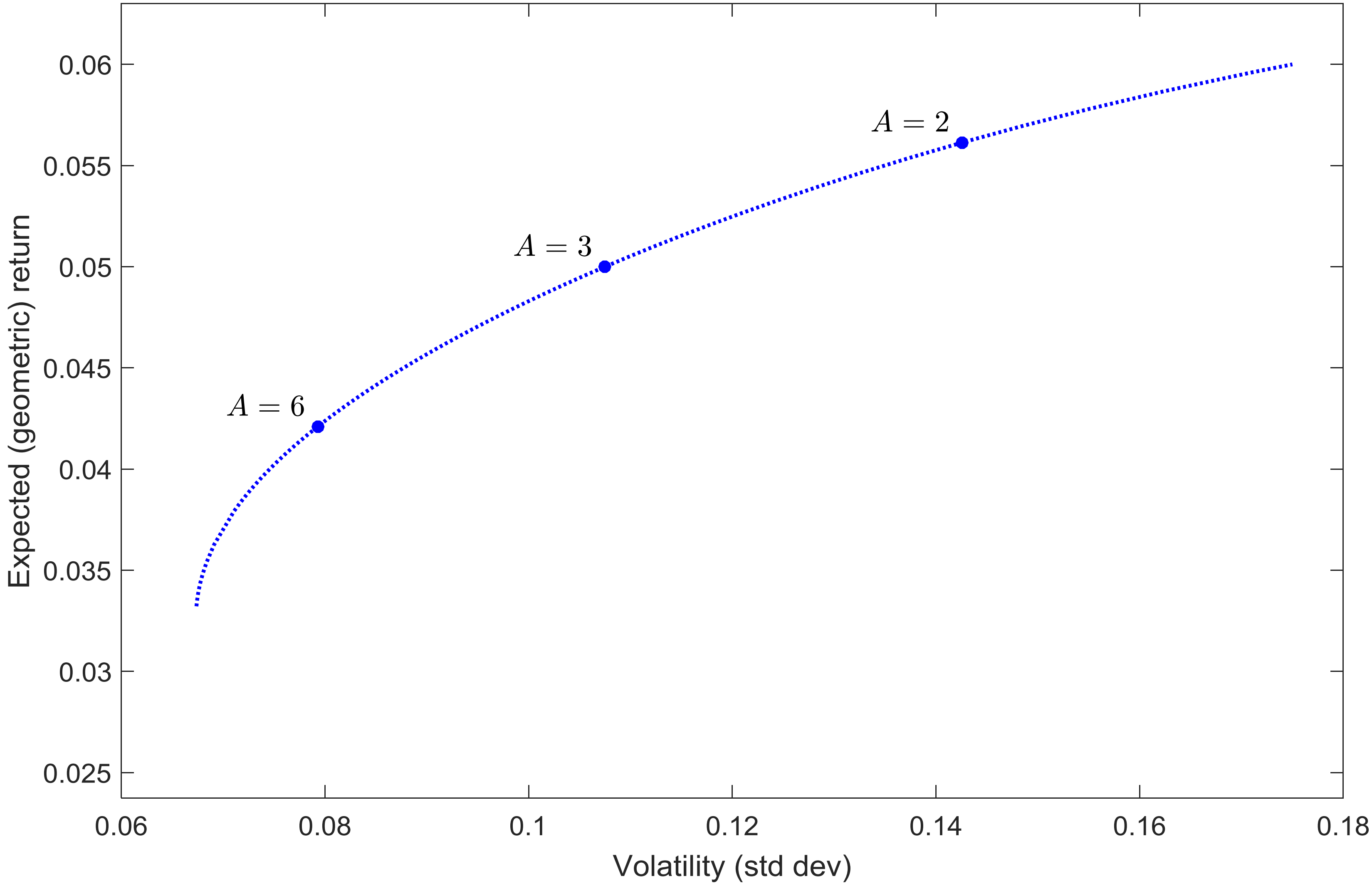
Power utility weights

A	Weight
6	37%
5	41%
4	48%
3	59%
2	81%
0.5	103%

Assume return correlation of zero between stocks and bonds

Where on the efficient frontier?

What's your target compounding rate and horizon?



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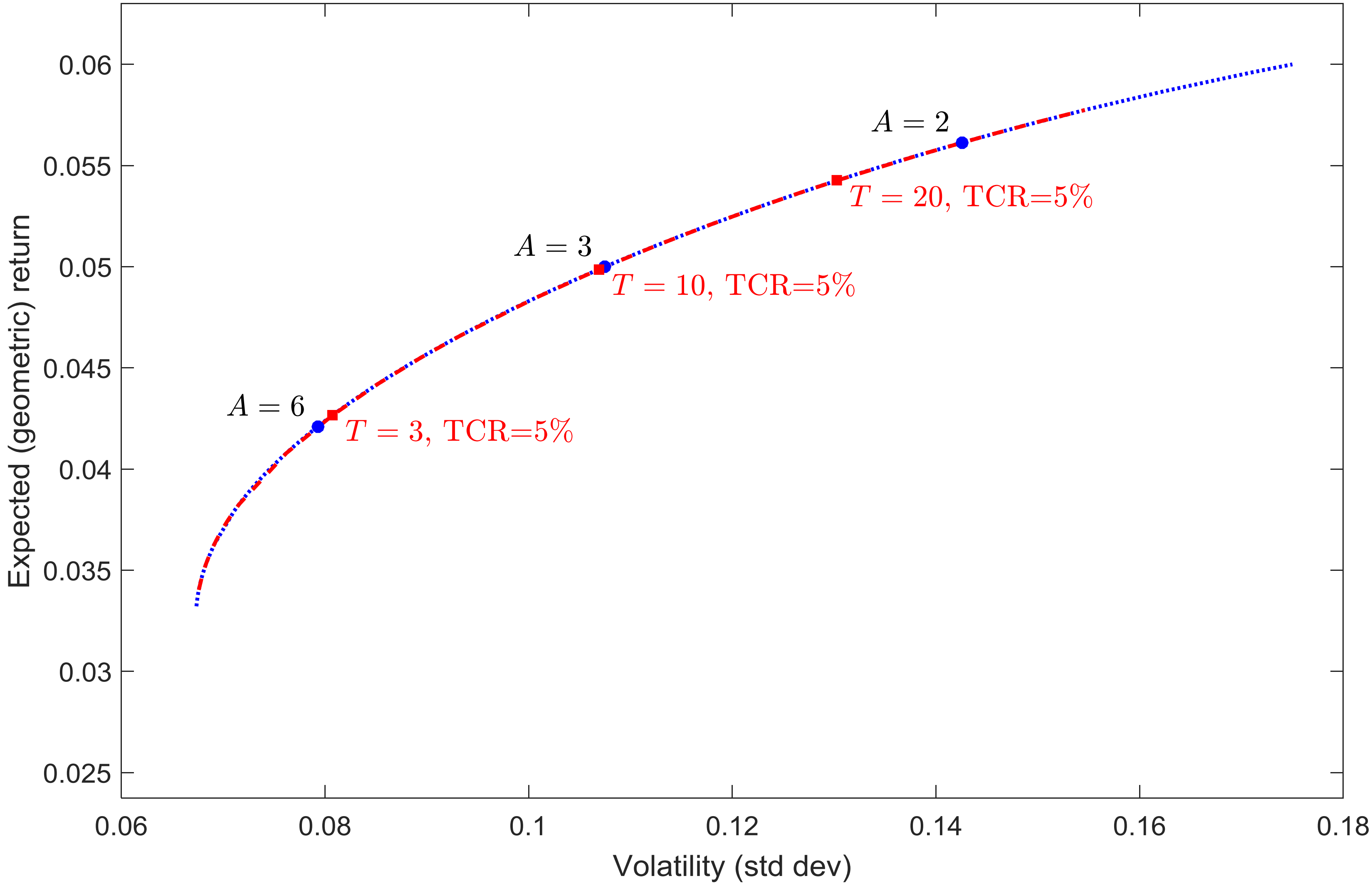

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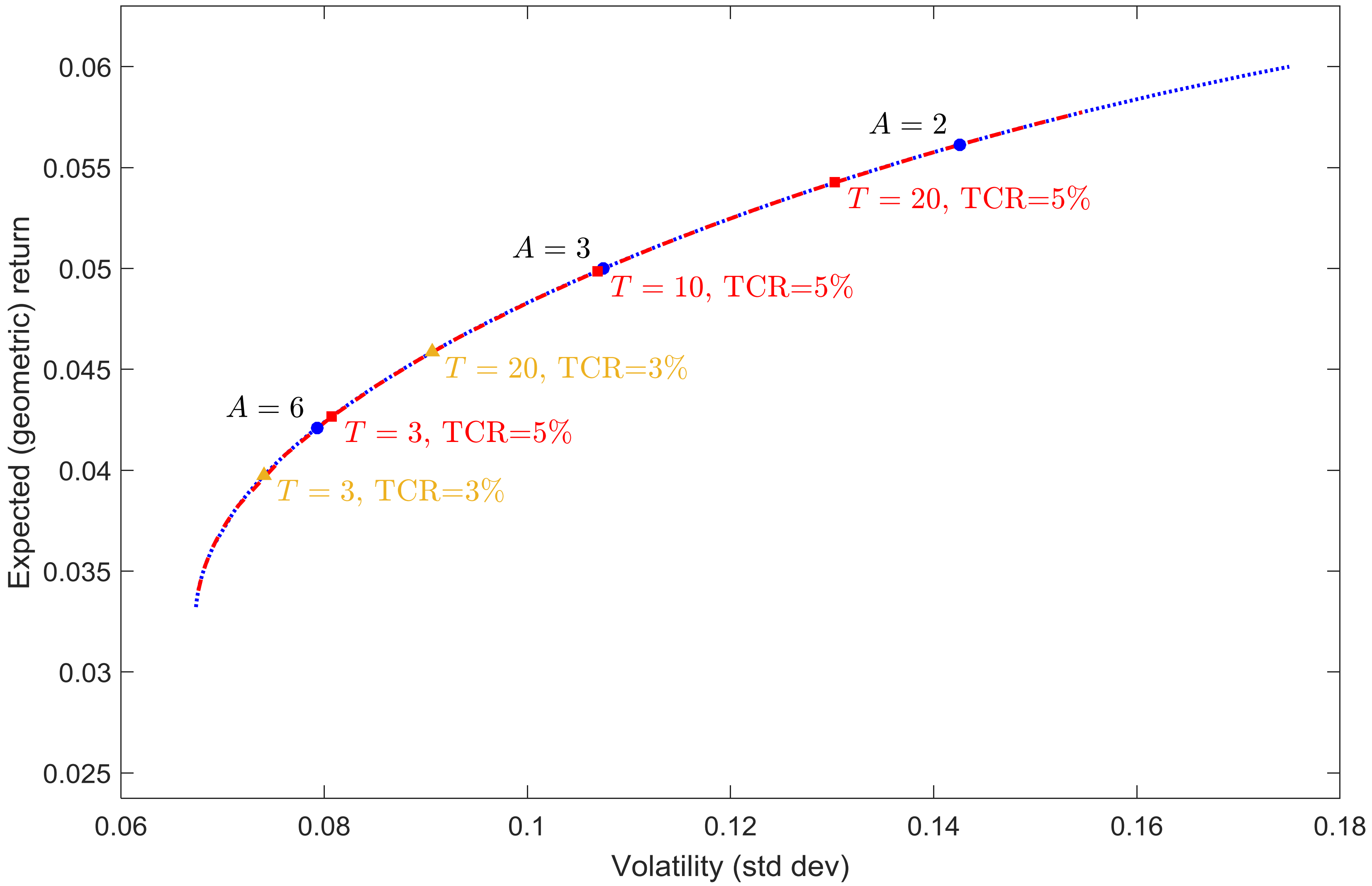

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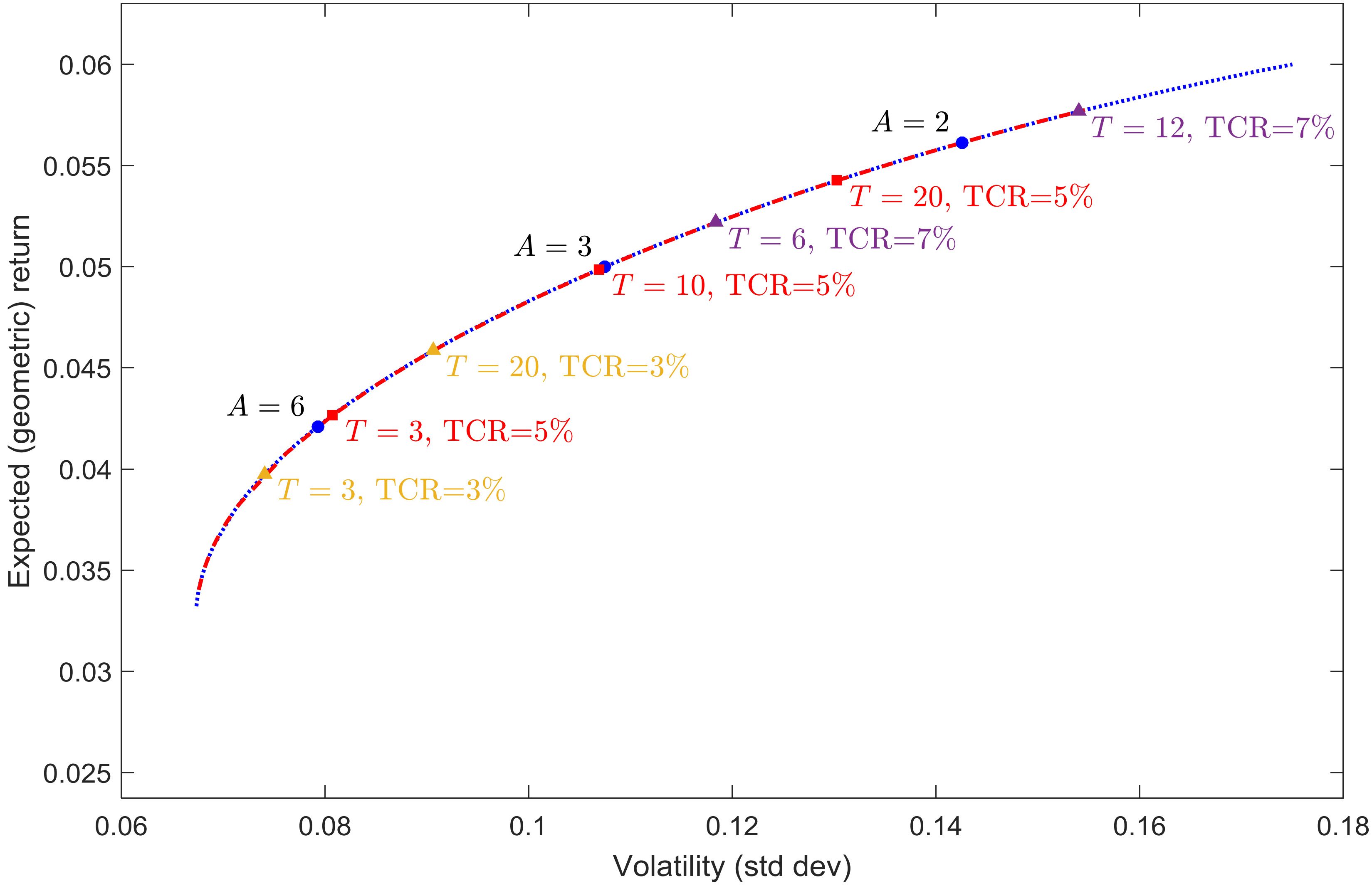
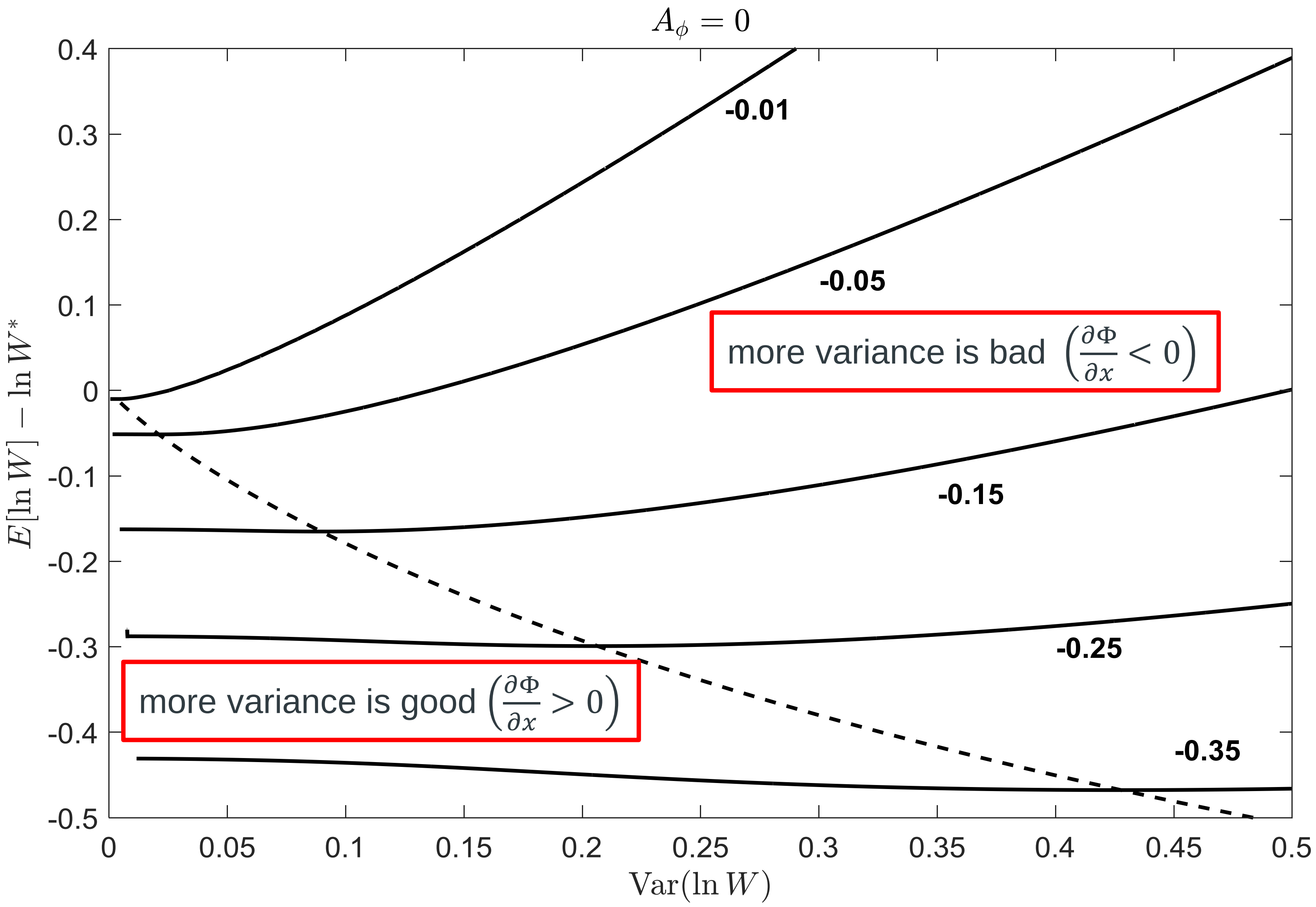


Expected shortfall is “mostly” M-V efficient

$\Phi(x, y)$ is a universal function of $x = \mathrm{Var}\,(\ln W)$ and $y = E[\ln W] - \ln W^*$

$\frac{\partial \Phi}{\partial y} > 0$



more variance is bad $\left(\frac{\partial \Phi}{\partial x} < 0\right)$

more variance is good $\left(\frac{\partial \Phi}{\partial x} > 0\right)$

An example – stock-bond allocation

Optimize pure shortfall utility function: $\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0$

Stock weights as a function of target compounding rate and horizon

Target Compounding Rate	T=1 yr	T=3	T=5	T=7	T=10	T=20
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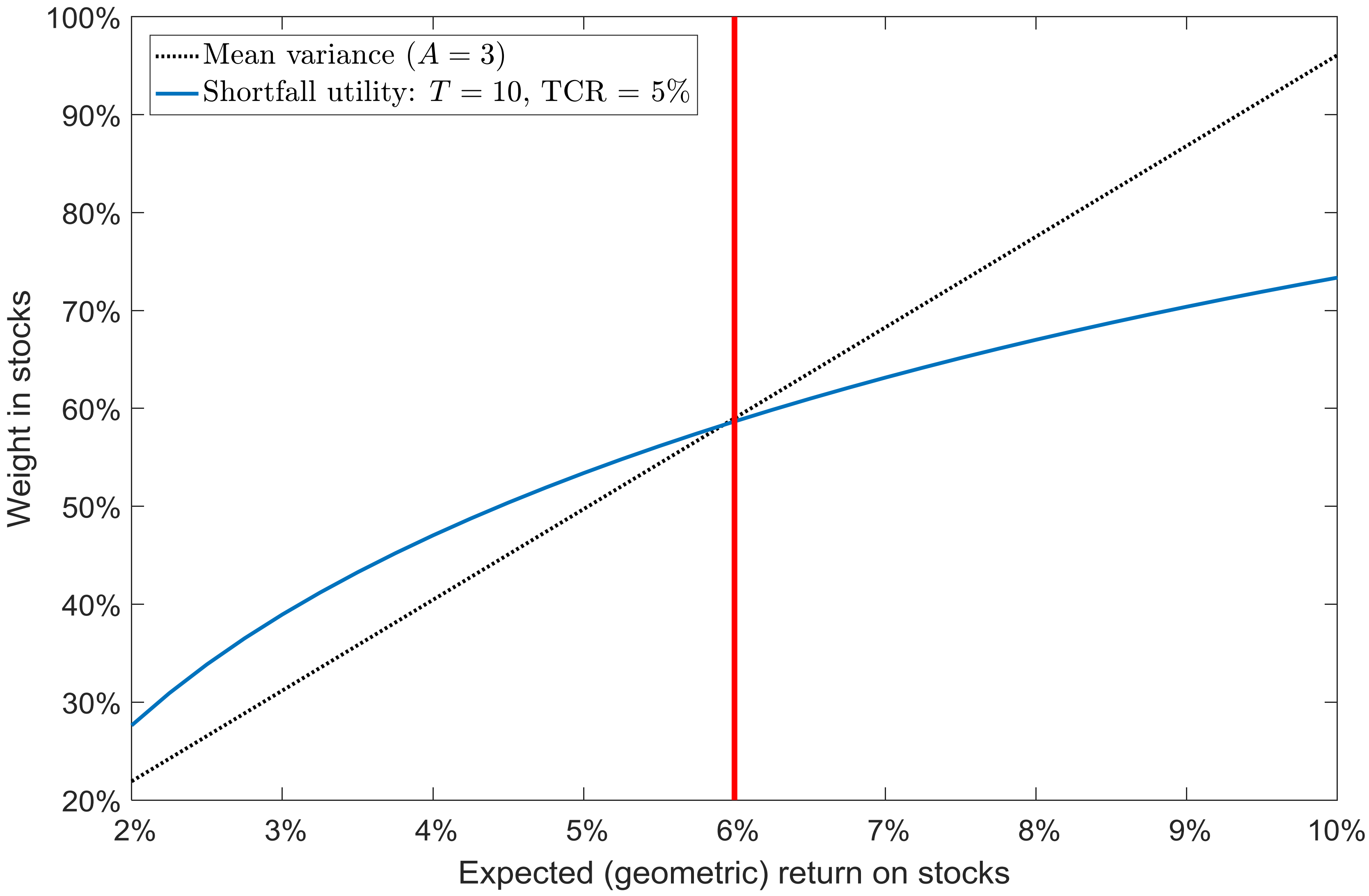
Sensitivity to expected returns – iid case



TCR: target compounding rate

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What do you need and when do you need it?

Two period binomial model

A simple example

- Contrast target compounding rate and target wealth

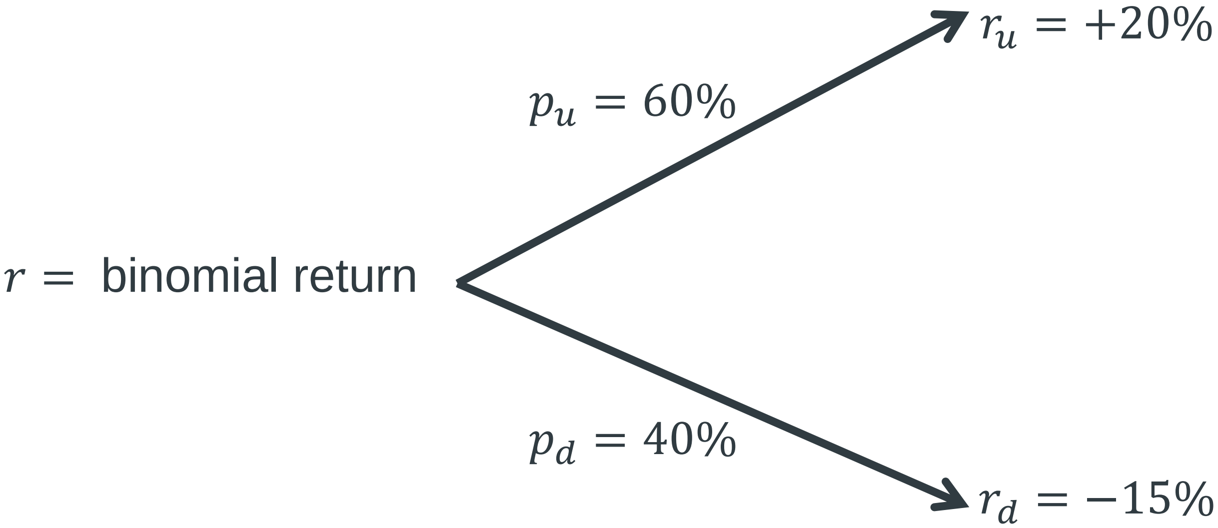
- Tactical becomes strategic

- Optimization

Two period binomial model

A simple example

- Two assets
  - A store of wealth with zero return, zero volatility
  - A volatile asset with 6% expected real return, and 17% annualized vol



$r =$ binomial return

$p_u = 60\%$

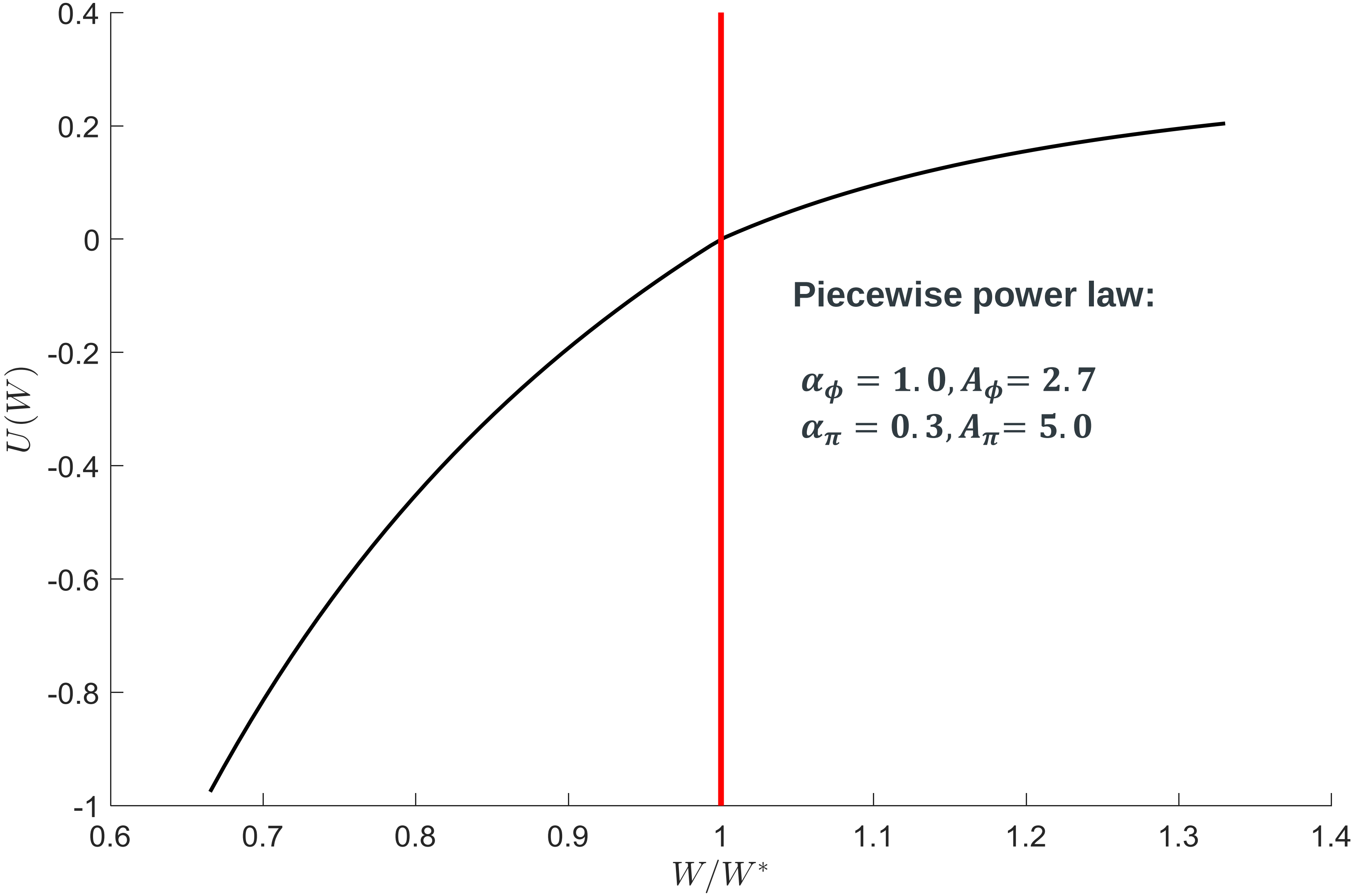
$r_u = +20\%$

$p_d = 40\%$

$r_d = -15\%$

How much do you care about not achieving the target?

Risk aversion at the target is a measure of how much you care



Piecewise power law:

$\boldsymbol{\alpha_\phi = 1.0, A_\phi = 2.7}$
$\boldsymbol{\alpha_\pi = 0.3, A_\pi = 5.0}$

Two period binomial model

A simple example to illustrate constant TCR vs. constant wealth target

- Two periods
  - Start with $W_0 = \$1$, target $W^* = \$1.13$
  - Invest for two periods

$$W_1 = W_0(1 + r_0 x_0) = 1 + r_0 x_0$$
$$W_2 = W_1(1 + r_1 x_1) = (1 + r_0 x_0)(1 + r_1 x_1)$$

- How much to invest in the volatile asset at $t = 0$?
  - Maximize expected utility at $t = 2$

$$\max_{x_0, x_1} \quad V(x_0, x_1)$$

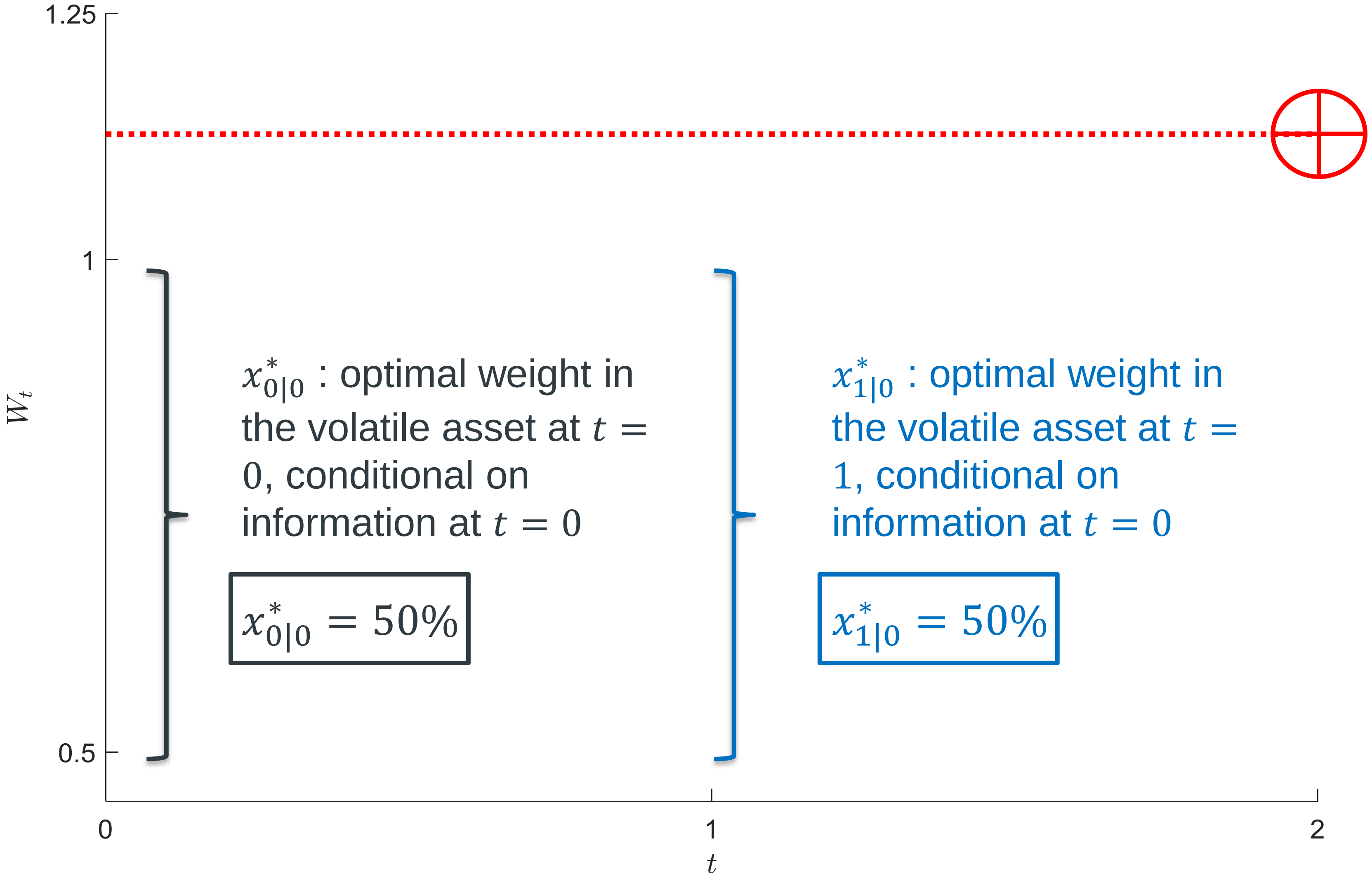
$$V(x_0, x_1) = p_u p_u U(W_2^{uu}) + p_u p_d U\left(W_2^{ud}\right) + p_d p_u U\left(W_2^{du}\right) + p_d p_d U\left(W_2^{dd}\right)$$

  - where $W_2^{uu} = (1 + r_u x_0)(1 + r_u x_1)$, $W_2^{ud} = (1 + r_u x_0)(1 + r_d x_1)$, …

QMα

Two period binomial model

Directly optimizing the two period objective generates a term structure of portfolios



$x^*_{0|0}$: optimal weight in the volatile asset at $t = 0$, conditional on information at $t = 0$

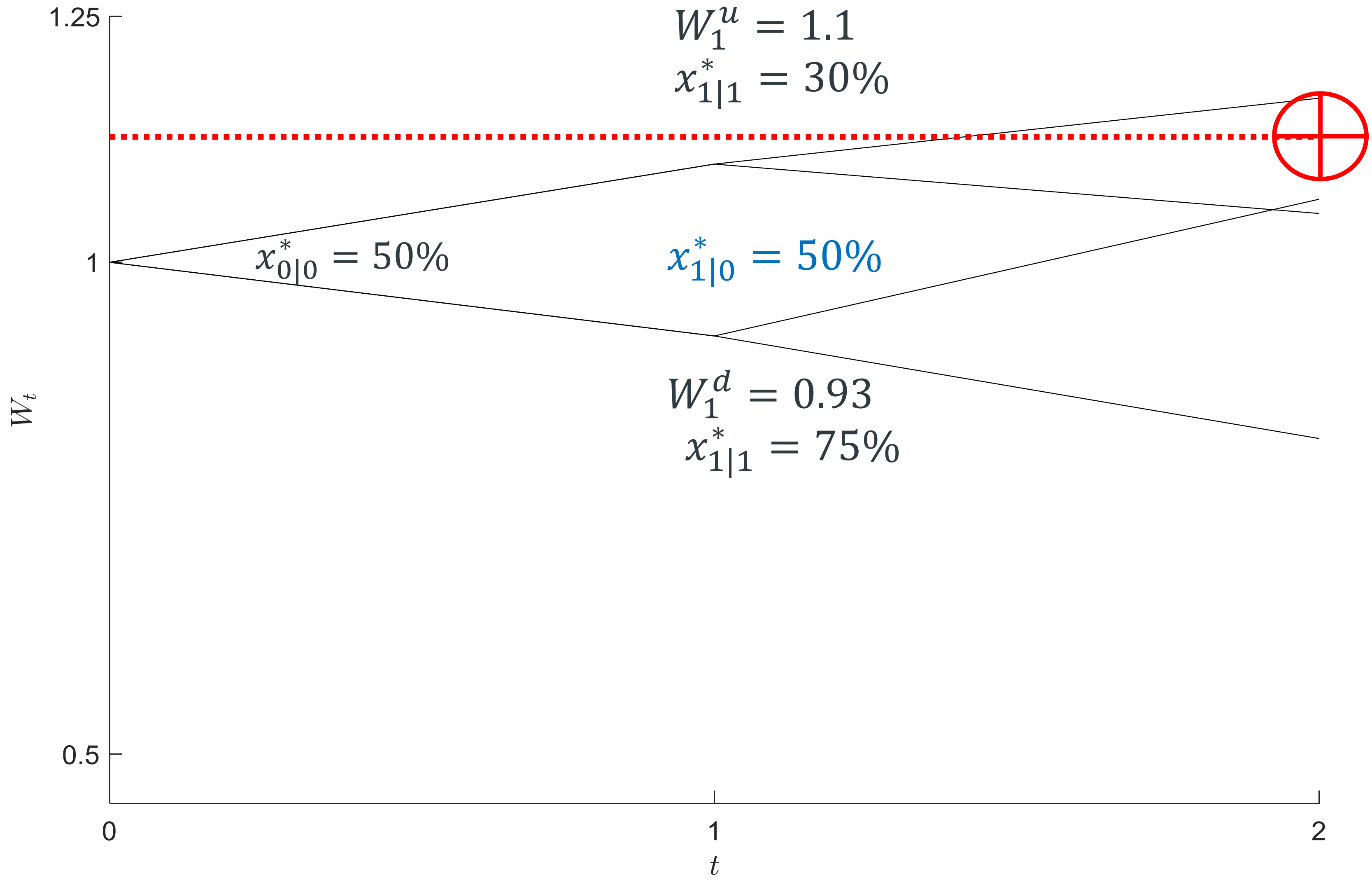
$\boxed{x^*_{0|0} = 50\%}$

$x^*_{1|0}$: optimal weight in the volatile asset at $t = 1$, conditional on information at $t = 0$

$\boxed{x^*_{1|0} = 50\%}$

Two period binomial model

$x^*_{1|0}$ accounts for the stochastic dynamics of wealth



Relationship to dynamic programming

Dynamic programming and the direct approach give the same answer for $x^*_{0|0}$

- Basic problem

$$\max_{x_0,x_1} E_0[U(W_2)]$$

$$W_2 = \underbrace{W_0(1+r_0x_0)}_{W_1}(1+r_1x_1)$$

- Rewrite the basic problem as

$$\max_{x_0} E_0\left[\max_{x_1} E_1[U(W_2)]\right]$$

- Dynamic programming works backwards, by

1. first solving $J_1(W_1) = \max_{x_1} E_1[U(W_2)]$ as a function of W_1
2. then solving $J_0(W_0) = \max_{x_0} E_0[J_1(W_1)]$ as a function of W_0

- Both the direct method and dynamic programming give the same answer for the optimal “current” weight $x^*_{0|0}$ at $t=0$, conditional on information at $t=0$

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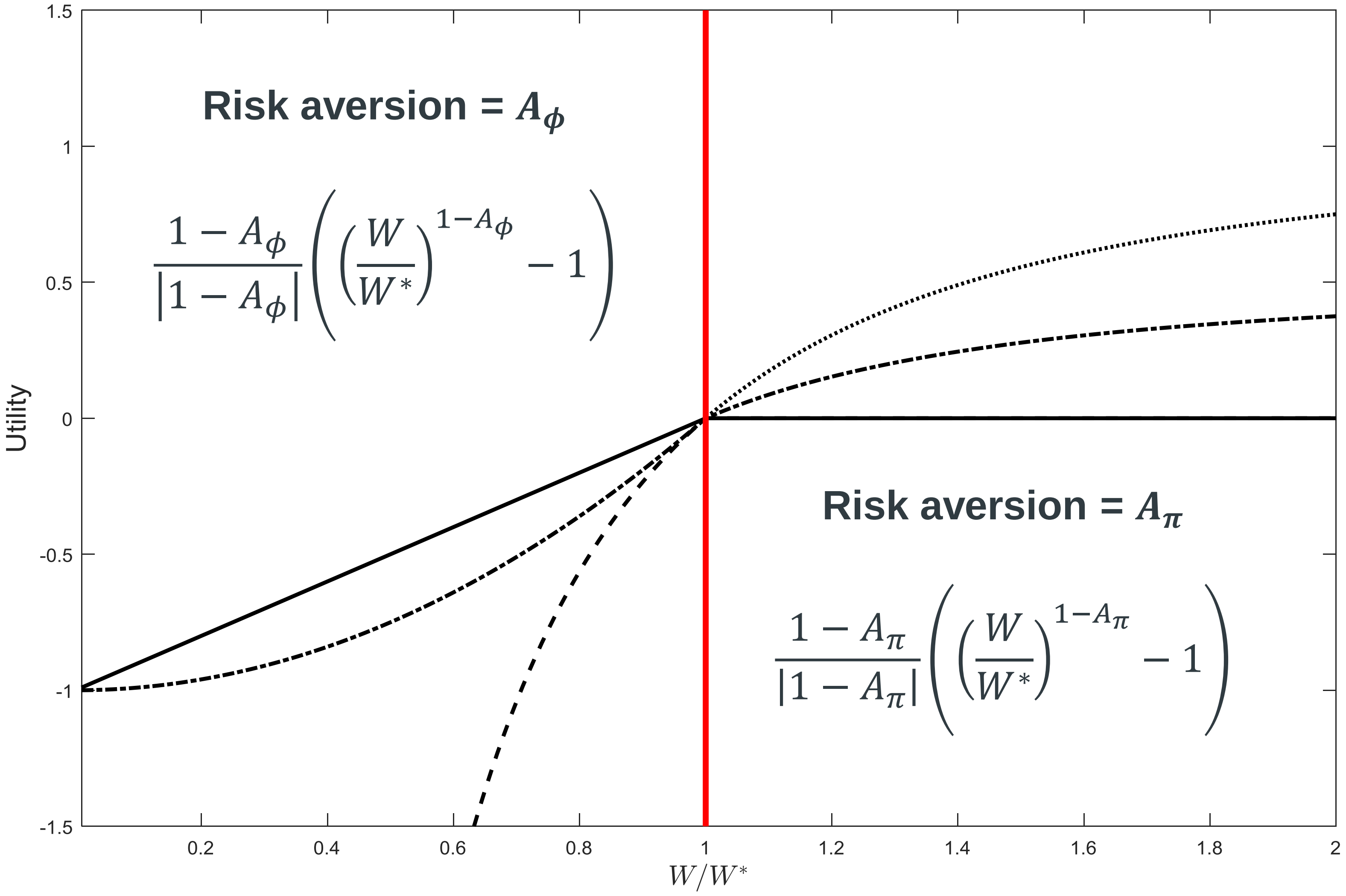
What do you need and when do you need it?

How much do you care about not achieving your target?

- Asymmetric preferences between shortfall and surplus reflect how much you care about not achieving your target
  - Can we characterize this asymmetry?

How much do you care about not achieving the target?

Risk aversion at the target is a measure of how much you care



Risk aversion = A_ϕ

$$\frac{1-A_\phi}{|1-A_\phi|}\left(\left(\frac{W}{W^*}\right)^{1-A_\phi}-1\right)$$

Risk aversion = A_π

$$\frac{1-A_\pi}{|1-A_\pi|}\left(\left(\frac{W}{W^*}\right)^{1-A_\pi}-1\right)$$

Risk premium and risk aversion

How much are you willing to pay to avoid a fair gamble?

- When wealth is either ***below*** or ***above*** the target

risk premium ~ A_2 x variance of the gamble

- ➢ $A_2 = \begin{cases} A_\phi, \text{if } W < W^* \\ A_\pi, \text{if } W > W^* \end{cases}$ is the coefficient of second order risk aversion

- When wealth is ***at*** the target

risk premium ~ A_1 x std dev of the gamble

  - A_1 is the coefficient of first order risk aversion
  - When you care about shortfall more than surplus, i.e. $\alpha_\phi|1 - A_\phi| > \alpha_\pi|1 - A_\pi|$

$$A_1 = 1 - \frac{\alpha_\pi|1 - A_\pi|}{\alpha_\phi|1 - A_\phi|}$$

First order risk aversion

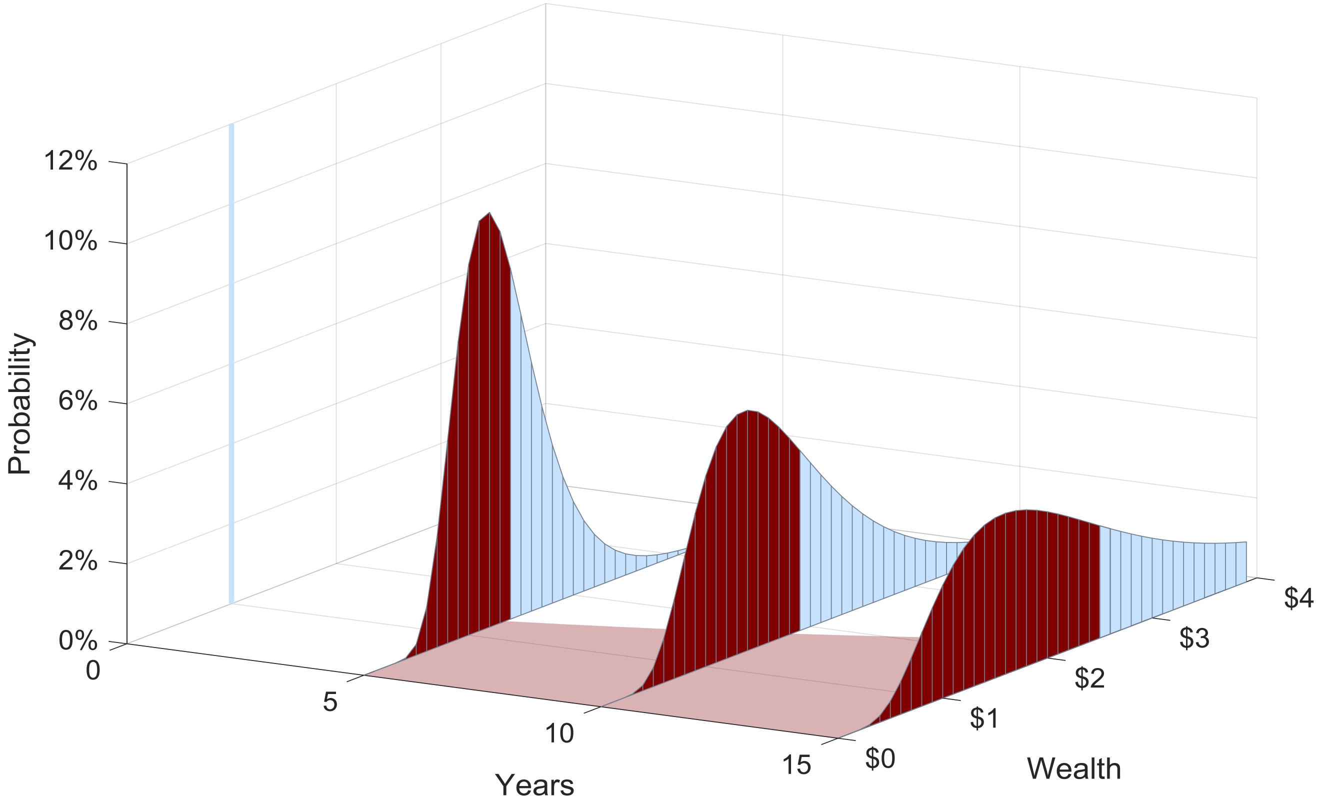
Is a rough measure of how much you care about not achieving the target

$$A_1 = 1 - \frac{\alpha_\pi |1 - A_\pi|}{\alpha_\phi |1 - A_\phi|} \quad \text{if} \quad \underbrace{\alpha_\phi |1 - A_\phi| > \alpha_\pi |1 - A_\pi|}_{\text{Shortfall matters more than surplus}}$$

	α_ϕ	A_ϕ	α_π	A_π	A_1
Pure power law	α	A	α	A	0
Pure shortfall	α_ϕ	A_ϕ	0	-	1
Binomial model example	1	2.7	0.3	5	0.29

Extensions

- Mean reverting expected returns (see SSRN paper)
- Shortfall over a range of times rather than a single point
- Multiple wealth targets



Summary

Investment risk is not having what you need when you need it

Specifying

1. Target wealth or target compounding rate
2. Investment horizon
3. Preferences for shortfall and surplus

leads to a framework that addresses

1. *What* do you need/desire?
2. How much do you care about not achieving your need/desire?
3. *What* do you have?
4. *When* do you need/desire it?

Horizon comes to the fore

Appendix

Expected utility – shortfall

Objective function depends on “mean”, “variance”, and preferences

- Key point

$$\Phi\Big(\underbrace{\overbrace{E[\ln\ W]}^{\text{Expected log wealth}},\overbrace{Var(\ln W)}^{\text{Variance of log wealth}}}_{\text{Depend on how you invest}}\ ;\ \underbrace{W^*,\ A_\phi}_{\text{Investor preferences}}\Big)$$

- The explicit formula for expected shortfall utility is

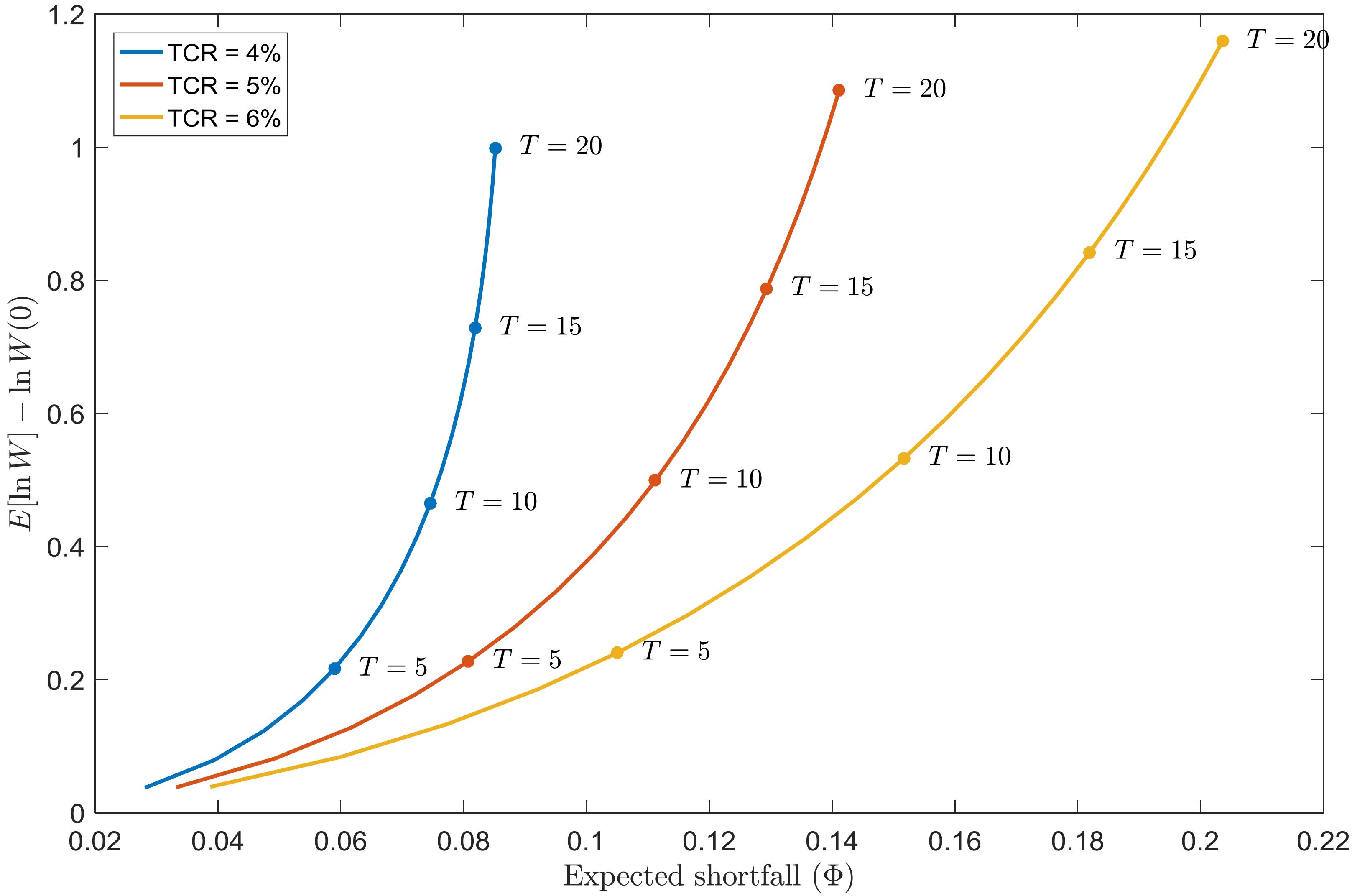
$$\Phi = \frac{1-A_\phi}{\left|1-A_\phi\right|}\left\{-N(z_1) + e^{(1-A_\phi)(E[\ln W]-\ln\ W^*)+\frac{1}{2}(1-A_\phi)^2 Var(\ln W)}\ N\big(z_2(A_\phi)\big)\right\}$$

  - Definitions
    - $N(\cdot)$ = standard cumulative normal
    - $z_1 = \frac{\ln\ W^* - E[\ln W]}{\sqrt{Var(\ln W)}}, \quad z_2(A_\phi) = z_1 - \big(1-A_\phi\big)\sqrt{Var(\ln W)}$

- Resembles price of a European put option for stock price = strike price = 1
  - $P = -N(-d_1) + e^{-rt}N(-d_2)$
  - But results from straightforward evaluation of the expectation integral, there are no replicating portfolios or no arbitrage assumptions

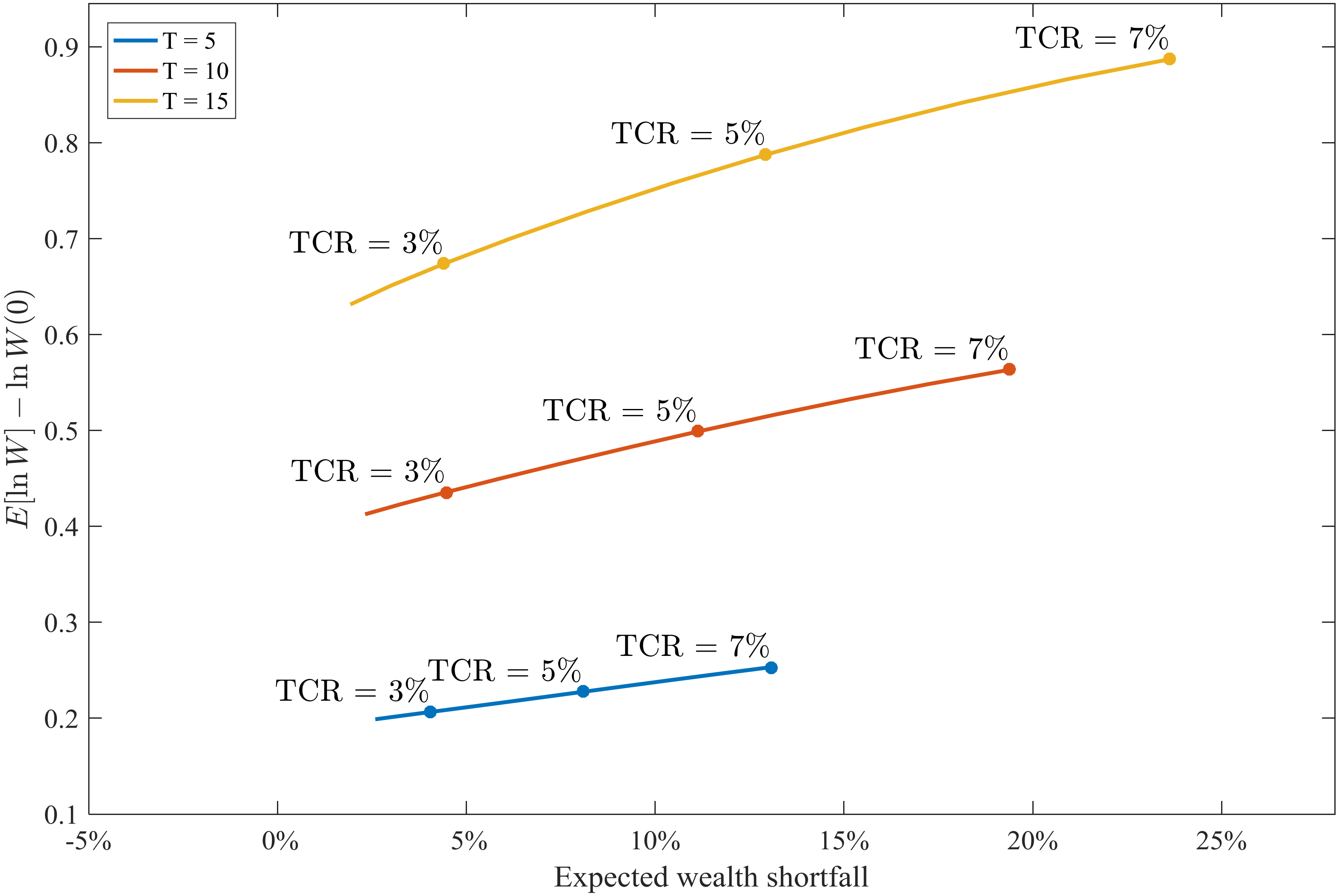
Reward and investment risk

Multiple target compounding rates



Reward and *investment* risk

Vary target compounding rates



Notes to Disclosure

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