Multi-Asset Class Risk Models

Overcoming the Curse of Dimensionality

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Bloomberg
Outline

- Motivation and Overview

- Candidate Models

- Evaluating the Accuracy of Correlation Forecasts
  - Factor-pair portfolios

- Evaluating the Quality of Optimized Portfolios
  - Out-of-sample volatility
  - Turnover
  - Leverage

- Ranking the Candidate Models

- MAC2 versus MAC1 Comparison

- Summary
Motivation and Overview
Multi-Asset Factor Covariance Matrices

- Portfolios may have exposure to multiple asset classes
- Each asset class is composed of multiple local markets
- Each local market is explained by many local factors

To obtain accurate risk forecasts for any portfolio requires a covariance matrix that combines all of the local factors

\[
F = \begin{bmatrix}
F_{EQ} & F_{EQ} & \cdots & F_{EQ} \\
F_{FI} & F_{FI} & \cdots & F_{FI} \\
\vdots & \vdots & \ddots & \vdots \\
F_{FX} & F_{FX} & \cdots & F_{FX}
\end{bmatrix}
\]

\[
F_{EQ} = \begin{bmatrix}
F_{USA} & F_{USA} & \cdots & F_{USA} \\
F_{JAP} & F_{JAP} & \cdots & F_{JAP} \\
\vdots & \vdots & \ddots & \vdots \\
F_{EUR} & F_{EUR} & \cdots & F_{EUR}
\end{bmatrix}
\]
The “Curse of Dimensionality”

- Forecasting accuracy requires a detailed factor structure spanning all markets and asset classes
  - Bloomberg MAC covariance matrix contains nearly 2000 factors
- Portfolio construction demands a robust covariance matrix
  - Risk model should not identify spurious hedges that fail out-of-sample
- With fewer observations than factors \((T<K)\), sample covariance matrix contains one or more “zero eigenvalues”
  - Leads to spurious prediction of “riskless” portfolios
- This feature makes the sample covariance matrix unsuitable for portfolio construction

Special methods are required to simultaneously provide:
1. Accurate volatility forecasts (risk management)
2. Robust risk models (portfolio construction)
Case Study: The Perils of Non-Robust Models

- Take the largest 100 US equities as of 16-Sept-2015, with complete return history to 13-Jan-1999
- Estimate family of asset covariance matrices using EWMA with a variable half-life
- Each week, construct the minimum-volatility fully invested portfolio
- Bias statistic represents ratio of realized risk to forecast risk
- Risk forecasts become increasingly poor as the half-life is shortened
Out-of-Sample Volatility

- Mean-variance optimization produces a portfolio with the minimum \textit{ex ante} volatility for a given level of factor exposure.
- If all stocks have the same expected return, the minimum volatility fully invested portfolio has the maximum Sharpe ratio.

Realized volatility increases as half-life parameter shortens:
- Equal-weighted portfolio had realized volatility of 16.5 percent.
- For HL<25w, optimization actually led to \textit{increased} portfolio volatility.
Leverage and Turnover

- Portfolio leverage increased sharply with shorter half-life
  - Half-life of 10 weeks produced a leverage of 10
  - Resulting portfolio is 550% long and 450% short

- Turnover rises dramatically with shorter half-life
  - 10-week half life produced more than 200% weekly turnover
Candidate Models
Separating Volatilities and Correlations

- Divide the task of constructing a factor covariance matrix into two parts:
  - Estimate the factor volatilities
  - Estimate the factor correlation matrix

- Factor volatilities are typically estimated using a relatively short half-life parameter (i.e., responsive forecasts)

- Factor correlations typically use longer half-life parameters
  - Reduces noise in the correlation matrix
  - Produces accurate risk forecasts

- The factor covariance matrix is easily reconstructed:
  \[ F_{jk} = \rho_{jk} \sigma_j \sigma_k \]

- Present study focuses on comparing the model quality of correlation matrices for the equity block
Sample Correlation Matrix

- Sample correlation matrix possesses many attractive properties:
  - Provides arguably the best estimate for any pairwise correlation
    - Best Linear Unbiased Estimate (BLUE) under standard econometric assumptions
  - Gives intuitive and transparent estimates, since it is based on the “textbook” definition of correlation coefficient
  - Produces accurate risk forecasts for most portfolios (with the notable exception of optimized portfolios)

- Sample correlation matrix also possesses an “Achilles heel”:
  - If there are $K$ factors and $T$ periods, then sample correlation matrix contains zero eigenvalues (i.e., rank-deficient matrix) whenever $T < K$
  - Rank-deficient matrices predict the existence of “phantom” riskless portfolios that do not exist in reality
  - Sample correlation is not robust for portfolio construction

Objectives: (a) correlation estimates should closely mimic the sample, and (b) provide robust forecasts for portfolio construction purposes
Other Techniques for Estimating Correlations

- Principal Component Analysis (PCA)
  - Statistical technique to extract global factors from the data
  - Assume a small number of global factors (principal components) fully capture correlations of local factors (i.e., uncorrelated residuals)

- Random Matrix Theory (RMT)
  - Statistical technique similar to PCA (factors extracted from data)
  - Eigenvalues beyond a cutoff point are simply averaged

- Time-series Approach
  - Specify “global” factor returns to explain “local” factor correlations
  - Estimate the exposures by time-series regression

- Eigen-adjustment Method
  - Eigenvalues of sample correlation matrix are systematically biased
  - Adjust the eigenvalues to remove biases

Blended Correlation Matrices

- Ledoit and Wolf (2003) showed that blending the sample covariance matrix with a one-factor model yielded optimized fully invested portfolios with lower out-of-sample volatility.

- Blend sample correlation (using weight $w$) with PCA correlation using $J$ principal components derived from $K$ local factors.

- Specify number of PCA factors by parameter $\mu$, where $J = \mu K$.

- Two-parameter model for correlation matrix:

  $$ C_B(\mu, w) = wC_0 + (1 - w)C_P(\mu) $$

- Optimal blending parameters are determined empirically.

- Technique represents the new Bloomberg methodology.

Adjusted Correlation Matrices

- Local models provide our “best” estimates of the correlation matrices for the diagonal blocks
- Global model is used to estimate the off-diagonal blocks
- Diagonal blocks of the global model differ from the correlation matrices obtained from the local models

- Integrated model is formed by “adjusting” the global model to replicate the local models along the diagonal blocks
Evaluating the Accuracy of Correlation Forecasts
Measuring Biases in Risk Forecasts

- Bias statistic represents the ratio of forecast risk to realized risk
  \[ z_{nt} = \frac{r_{nt}}{\sigma_{nt}} \rightarrow B_n = \text{std}(z_{nt}) \rightarrow \bar{B} = \frac{1}{N} \sum_n B_n \]

- If the risk forecasts are *exactly correct*, the expected value of the bias statistic is precisely equal to 1
- If the risk forecasts are *unbiased* but noisy, the expected value of the bias statistic is *slightly greater* than 1
- Example: suppose we over-forecast volatility by 10% half of the time, and under-forecast by 10% half the time
  \[
  E[B] = \sqrt{\frac{1}{2} \left( \frac{1}{0.9} \right)^2 + \frac{1}{2} \left( \frac{1}{1.1} \right)^2} = 1.02
  \]
- Typical bias statistic for unbiased risk forecasts is about 1.03
Factor-Pair Portfolios

- Construct test portfolios capable of resolving minor differences in volatility forecasts due to differences in correlations
- Consider factor-pair portfolios
  \[ R = f_1 + wf_2 \rightarrow \sigma^2 = \sigma_1^2 + w^2 \sigma_2^2 + 2 \rho w \sigma_1 \sigma_2 \]
- Solve for the weight \( w \) that maximizes the percentage of risk due to the off-diagonal correlation
- Solution is given by
  \[ w = \pm \left( \frac{\sigma_1}{\sigma_2} \right) \]
- Portfolio volatility
  \[ \sigma = \sqrt{2 \sigma_1 (1 - |\rho|)}^{1/2} \]
Description of Study

- Sample period contains 713 weeks (03-Jan-01 to 27-Aug-14)
- Model contains $K=319$ factors spanning nine equity blocks
- Evaluate accuracy of correlations using factor-pair portfolios

Parameters used in Study:

- Use $T=200$ weeks (equal weighted) as estimation window
- For PCA, RMT, and blended matrices
  - Use $\mu_L=0.25$ for local blocks
  - Use $\mu_G=0.10$ for global block
- For blended correlation matrices
  - Assign 80% weight to the sample ($w=0.8$) for local blocks
  - Assign 20% weight to the sample ($w=0.2$) for global blocks
- Blending parameter selection criteria:
  - Small deviation from the sample correlation
  - Low realized volatility of optimized portfolios
Correlation Scatterplots (Diagonal Blocks)

- Local Blended provides a near “perfect fit” to the sample
- Eigen method and Local RMT exhibit systematic biases

- Compute B-stats for all factor-pairs with mean sample correlation above 0.40 (292 portfolios)
- Eigen method and Local RMT exhibit biases
- Sample and Local Blended are near ideal value

<table>
<thead>
<tr>
<th>Model</th>
<th>Bias Stats</th>
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<td>Sample</td>
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<tr>
<td>Local Blended</td>
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<td>Eigen method</td>
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<tr>
<td>Local RMT</td>
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</table>
Correlation Scatterplots (Off-Diagonal Blocks)

- Global Blended provides an excellent fit to the sample
- Time Series and Global RMT exhibit systematic biases

- Compute B-stats for all factor-pairs with mean sample correlation above 0.50 (163 portfolios)
- Time Series and Global RMT exhibit biases
- Sample and Global Blended are near ideal value

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<td>Sample</td>
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<tr>
<td>Global Blended</td>
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<td>Time Series</td>
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<tr>
<td>Global RMT</td>
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Evaluating the Quality of Optimized Portfolios
Quality of Optimized Portfolios

- Portfolio optimization typically represents the most demanding task for any risk model (the “acid” test)
- Optimized portfolios have the maximum possible \textit{ex ante} information ratio
- This implies that optimized unit-exposure portfolios have the minimum volatility

\[ w^A_k = \frac{\Omega_A^{-1} \alpha_k}{\alpha_k' \Omega_A^{-1} \alpha_k} \quad \text{Optimal portfolio (Model A)} \]

- Define the mean volatility ratio for Model A

\[ \nu_A = \frac{1}{K} \sum_k \frac{\sigma^A_k}{\sigma^{Ref}_k} \quad \text{Model with lowest volatility ratio wins} \]

- Construct optimal portfolios for each of \( K=319 \) factors and rebalance on a weekly basis (Jan-2001 to Aug-2014)
Portfolio Optimization (Ex Ante)

- Decompose optimal portfolio into alpha and hedge portfolios:
  \[ w = \frac{\Omega^{-1}\alpha}{\alpha'\Omega^{-1}\alpha} \equiv \alpha + h \]

- Hedge portfolio is uncorrelated with the optimal portfolio
  \[ h'\Omega w = 0 \quad \text{Property 1} \]

- The hedge portfolio has zero alpha
  \[ h'\alpha = 0 \quad \text{Property 2} \]

- Hedge portfolio is negatively correlated with the alpha portfolio
  \[ \frac{h'\Omega \alpha}{\sigma_h \sigma_\alpha} < 0 \quad \text{Property 3} \]

Hedge portfolio reduces portfolio risk without changing the expected return.
Geometry of Portfolio Optimization (*Ex Ante*)

- Hedge portfolio is uncorrelated with optimal portfolio
  \[ \sigma^2_p = \sigma^2_\alpha - \sigma^2_h \]  
  Portfolio Variance

- Let \( \rho_{\alpha h} \) denote the predicted correlation between \( \alpha \) and \( h \)

- The magnitude of the correlation determines quality of hedge

- Optimal position in hedge portfolio:
  \[ \sigma_h = \sigma_\alpha |\rho_{\alpha h}| \]

\[ \rho_{\alpha h} = \cos(\theta) \]
Potential Pitfalls of Optimization

- Optimization leads to superior *ex ante* performance, but is no guarantee of improvement *ex post*

- Estimation error within the covariance matrix represents a potential pitfall in portfolio optimization

- Estimation error in the correlation:
  - Risk models “paint an overly rosy picture” of the correlation between the alpha and hedge portfolios

- Estimation error in the volatility:
  - Risk models may misestimate the volatility of the hedge portfolio

- Estimation error gives rise to several detrimental effects:
  - Underestimation of risk of optimized portfolios
  - Higher out-of-sample volatility of optimized portfolios
  - Positive realized correlation between optimized and hedge portfolios
Estimation Error in the Correlation

- Suppose that we estimate the volatility of the hedge portfolio correctly
- However, suppose that we over-estimate magnitude of correlation

**Side Effects:**

1) Optimized portfolio has high volatility out-of-sample
2) Hedge portfolio is positively correlated with optimized portfolio
**Biases in Correlation Forecasts (Within Block)**

- Compute correlations between alpha and hedge portfolios
- All models systematically over-predicted correlations

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<tr>
<th>Model</th>
<th>Ex Ante</th>
<th>Ex Post</th>
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<td>Sample</td>
<td>-0.85</td>
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<tr>
<td>Global PCA</td>
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<td>-0.51</td>
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<tr>
<td>Local Blended</td>
<td>-0.74</td>
<td>-0.62</td>
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</table>

- Local blended hedge portfolio nearly as effective as sample hedge portfolio (ex post)
- Bias was smaller for local blended model than for sample, allowing better sizing of position
Objectives in Portfolio Optimization

- Low out-of-sample volatility for optimized portfolios
  - Measure using volatility ratio
- Accurate risk forecasts for optimized portfolios
  - Measure using bias stats or Q-stats
- Efficient *ex post* allocation of the risk budget
  - Realized risk should align with expected returns
- Low factor leverage
  \[ L_t = \sum_k |X_{k,t}| \rightarrow \bar{L} = \frac{1}{T} \sum_t L_t \]
- Low factor turnover
  \[ TO_t = \sum_k |X_{k,t+1} - X_{k,t}| \rightarrow \overline{TO} = \frac{1}{T} \sum_t TO_t \]
Optimized Factor Portfolios (Diagonal Blocks)

- Compute averages over all 319 optimized factor portfolios
- Allow hedging using only factors within the same block

<table>
<thead>
<tr>
<th>Model</th>
<th>B-stats</th>
<th>Q-stats</th>
<th>Vol Ratio</th>
<th>Leverage</th>
<th>Turnover</th>
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<tbody>
<tr>
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<td>0.955</td>
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<td>Global RMT</td>
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<td>2.471</td>
<td>1.072</td>
<td>2.583</td>
<td>0.088</td>
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<td>Eigen-method</td>
<td>1.376</td>
<td>2.891</td>
<td>0.882</td>
<td>12.392</td>
<td>0.523</td>
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<td>Local Blended</td>
<td>1.204</td>
<td>2.569</td>
<td>0.903</td>
<td>6.861</td>
<td>0.352</td>
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</tbody>
</table>

- Sample correlation underpredicted risk by factor of 2
- Sample and eigen-method had highest turnover and leverage
- Local blended model scored well across measures
Optimized Factor Portfolios (Across Blocks)

- Compute averages over all 319 optimized factor portfolios
- Allow hedging using local factors from different blocks

<table>
<thead>
<tr>
<th>Model</th>
<th>B-stats</th>
<th>Q-stats</th>
<th>Vol Ratio</th>
<th>Leverage</th>
<th>Turnover</th>
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<td>1.417</td>
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- Matrix adjustment replicates local models for diagonal blocks:
  - Allows for more effective hedging using factors within the same block
  - Reduces out-of-sample portfolio volatility
  - Generally increases portfolio leverage and turnover
Parameter Selection (US Equities)

- Leverage/Turnover minimized by few PC and low sample weight
- STD is minimized by taking many PC and high sample weight
- Volatility is minimized at intermediate values

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<td>0.70</td>
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<td>0.49</td>
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<td>0.80</td>
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<td>0.050</td>
<td>0.033</td>
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<tr>
<td>0.20</td>
<td>0.054</td>
<td>0.043</td>
<td>0.032</td>
<td>0.021</td>
<td>0.011</td>
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<tr>
<td>0.30</td>
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<td>0.036</td>
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<tr>
<td>0.40</td>
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<td>0.50</td>
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<td>0.023</td>
<td>0.017</td>
<td>0.012</td>
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<tr>
<td>0.60</td>
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<td>0.009</td>
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<td>0.70</td>
<td>0.016</td>
<td>0.013</td>
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<td>0.000</td>
<td>0.000</td>
<td></td>
</tr>
</tbody>
</table>
Volatility Ratio

- Create a Local PCA and Blended model for every local block
  - For Local PCA, vary number of principal components ($\mu_L$)
  - For Local Blended model, choose $\mu_L = 0.2$ and vary $w$

- Local PCA exhibited local minimum, but sample had even lower out-of-sample volatility

- Across block uses global PCA with local model as the target
Bias Statistics

- Local PCA model vastly under-predicts risk as the number of principal components approaches the maximum (i.e., $\mu \to 1$)
- Sample makes more accurate risk forecasts than the Local PCA model with many factors
- Even small blending ($w=0.8$) improves accuracy of risk forecasts for optimized portfolios

![Local PCA Model](image1)

![Local Blended ($\mu=0.2$)](image2)
Factor Leverage

- Compute mean factor leverage across factors and time

\[ L_t = \sum_k \left| X_{k,t} \right| \quad \rightarrow \quad \bar{L} = \frac{1}{T} \sum_t L_t \]

- For Local PCA, leverage increases sharply as \( \mu \rightarrow 1 \)
- For blended model, even small mixing (\( w=0.8 \)) is sufficient to greatly reduce mean factor leverage
**Factor Turnover**

- Compute mean factor turnover

\[ \bar{TO} = \frac{1}{T} \sum_{t} TO_t \]

- For Local PCA, turnover rises sharply with increasing \( \mu \)
- Blended model reduced factor turnover considerably
Ranking the Candidate Models
Ranking the Models within Local Markets

- Use two measures for forecast accuracy
- Use two measures for quality of optimized portfolios
- Convert into z-scores (positive scores are above average)
- Form composite z-score (equally weight four components)

<table>
<thead>
<tr>
<th>Model</th>
<th>Factor-Pairs Q-stats</th>
<th>Factor-Pairs Std</th>
<th>Optimized Factors Vol Ratio</th>
<th>Optimized Factors Turnover</th>
<th>Composite z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample</td>
<td>0.536</td>
<td>1.104</td>
<td>0.451</td>
<td>-1.722</td>
<td>0.184</td>
</tr>
<tr>
<td>Local PCA</td>
<td>0.254</td>
<td>-0.057</td>
<td>-0.706</td>
<td>0.042</td>
<td>-0.234</td>
</tr>
<tr>
<td>Global PCA</td>
<td>0.635</td>
<td>0.222</td>
<td>-0.243</td>
<td>-0.213</td>
<td>0.200</td>
</tr>
<tr>
<td>Local RMT</td>
<td>0.022</td>
<td>0.051</td>
<td>-0.630</td>
<td>0.563</td>
<td>0.003</td>
</tr>
<tr>
<td>Global RMT</td>
<td>-0.002</td>
<td>0.006</td>
<td>-0.327</td>
<td>0.181</td>
<td>-0.071</td>
</tr>
<tr>
<td>Time Series</td>
<td>-2.405</td>
<td>-2.228</td>
<td>-1.336</td>
<td>1.728</td>
<td>-2.118</td>
</tr>
<tr>
<td>Eigen-method</td>
<td>0.443</td>
<td>0.029</td>
<td>1.559</td>
<td>-0.782</td>
<td>0.624</td>
</tr>
<tr>
<td>Local Blended</td>
<td>0.517</td>
<td>0.872</td>
<td>1.233</td>
<td>0.203</td>
<td>1.411</td>
</tr>
</tbody>
</table>

New Bloomberg methodology performed above average on all four measures and earned the highest composite score
Ranking the Models across Multiple Markets

- New Bloomberg methodology
  - Scored above average on three of four measures
  - Produced highest composite z-score

<table>
<thead>
<tr>
<th>Model</th>
<th>Factor-Pairs</th>
<th>Optimized Factors</th>
<th>Composite z-score</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q-stats</td>
<td>Std</td>
<td>Vol Ratio</td>
</tr>
<tr>
<td>Global PCA</td>
<td>0.767</td>
<td>0.474</td>
<td>-0.850</td>
</tr>
<tr>
<td>Global RMT</td>
<td>-1.998</td>
<td>0.334</td>
<td>-0.933</td>
</tr>
<tr>
<td>Time Series</td>
<td>-1.008</td>
<td>-2.146</td>
<td>-1.327</td>
</tr>
<tr>
<td>Global Blended</td>
<td>0.820</td>
<td>0.717</td>
<td>-0.489</td>
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<tr>
<td>Global PCA (Adjusted)</td>
<td>0.415</td>
<td>0.400</td>
<td>1.000</td>
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<tr>
<td>Global RMT (Adjusted)</td>
<td>0.463</td>
<td>0.405</td>
<td>1.125</td>
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<tr>
<td>Time Series (Adjusted)</td>
<td>-0.044</td>
<td>-0.869</td>
<td>0.568</td>
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<tr>
<td>Global Blended (Adjusted)</td>
<td>0.585</td>
<td>0.684</td>
<td>0.905</td>
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</tbody>
</table>

- Unified methodology applied throughout estimation process facilitates implementation and comprehension of model
MAC2 versus MAC1 Comparison
Bloomberg MAC2 and MAC1 Models

- **MAC1** refers to the first-generation Bloomberg MAC model:
  - Computes diagonal blocks using RMT method with shrinkage
  - Computes off-diagonal blocks using the time-series method
  - For equities, “core” factors taken from global equity model
    - e.g., Japan autos is regressed on Japan factor and global auto factor
  - For other blocks, “core” factors are weighted average of local factors
    - e.g., Core factor for oil commodities is weighted average of Brent and WTI “shift”
  - Apply integration matrix to recover the diagonal blocks

- **MAC2** refers to the new Bloomberg MAC model:
  - Uses blended methodology for both diagonal and off-diagonal blocks
  - Applies integration matrix to recover local models on diagonal blocks

- **MAC1 and MAC2 models** use EWMA with same HL parameters:
  - 26 weeks for volatilities
  - 52 weeks for correlations
Diagonal Equity Blocks

- Make scatterplots of estimated correlations versus sample
- Example: US equity factors versus US equity factors

MAC2 estimates closely mimic the sample correlation
MAC1 estimates exhibit more scatter and some biases
Off-Diagonal Equity Blocks

- Make scatterplots of estimated correlations versus sample
- Example: US equity factors versus Japan equity factors

MAC2 estimates closely mimic the sample correlation
MAC1 estimates exhibit more scatter and some biases
US Equity versus US Fixed Income

- MAC2 estimates closely mimic the sample correlation
- MAC1 correlations exhibit considerable biases

Results suggest that the time-series method may not be effective at fully explaining correlations across asset classes
Cross Asset-Class Correlations versus Time

- Consider the correlation between the US energy factor (equity) and the crude-oil commodity factor (Brent shift)
- Plot predicted and realized correlations (52w HL)

- Blended approach captures the observed relationship very closely
- Time-series method systematically under-predicts correlation
- Suggests missing factors in time-series approach
Scenario Analysis

- What is the expected impact of a 20% drop in crude oil on the return of the S&P 500?

\[ E[R_P] = \sum_k X_k^P E[f_k] \]

- Shocked variables are propagated to other factors

\[ E[f_k] = \left( \frac{\rho_{kS} \sigma_k}{\sigma_S} \right) R_S \]

- Differences in correlations lead to differences in the expected returns of the propagated variables

![Crude Oil versus US Market](image.png)
Scenario Analysis Attribution

- MAC1 predicts drop of 2.36% for S&P 500
- MAC2 predicts drop of 3.61% for S&P 500
- MAC2 betas were generally larger due to higher correlations

### MAC1 Model

<table>
<thead>
<tr>
<th>Factor</th>
<th>Exposure</th>
<th>Beta</th>
<th>Propagation</th>
<th>Contribution</th>
</tr>
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<tbody>
<tr>
<td>US Market</td>
<td>1.000</td>
<td>0.120</td>
<td>-2.40%</td>
<td>-2.40%</td>
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<tr>
<td>Energy</td>
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<td>0.246</td>
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<td>Utilities</td>
<td>0.036</td>
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<td>1.81%</td>
<td>0.07%</td>
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<tr>
<td>Other Factors</td>
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<td>...</td>
<td>...</td>
<td>0.33%</td>
</tr>
<tr>
<td>Total</td>
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<td></td>
<td></td>
<td>-2.36%</td>
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### MAC2 Model

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<th>Propagation</th>
<th>Contribution</th>
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<tr>
<td>US Market</td>
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<td>0.192</td>
<td>-3.83%</td>
<td>-3.83%</td>
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<tr>
<td>Oil Exploration</td>
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<td>0.473</td>
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<tr>
<td>Utilities</td>
<td>0.036</td>
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<td>Other Factors</td>
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<td>-3.61%</td>
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Optimized Factor Portfolios (Across Blocks)

- For each of the 319 local equity factors, compute the volatility ratio between MAC2 and MAC1 for optimized factor portfolios

\[ v = \frac{1}{K} \sum_{k} \frac{\sigma_{k}^{\text{MAC2}}}{\sigma_{k}^{\text{MAC1}}} \]

- MAC2 Model produced lower volatility in more than 80 percent of portfolios
- The average volatility ratio was 0.89
- Similar results hold for within-block optimizations

Sample Period: 30-Mar-2005 to 27-Aug-2014
Summary

- Introduction of second-generation Bloomberg model (MAC2)
- Adopted “blank-slate” approach to select the best model among a broad set of candidate models
- New methodology:
  - Two-parameter model uses blended correlations at all estimation levels
  - Parameters are empirically determined
  - Generally, larger blocks assign smaller weight to sample correlation
  - Integration matrix is applied to recover local models on diagonal blocks
- New model closely mimics the sample correlation even across different asset classes (e.g., equity versus fixed income)
- New model guarantees full-rank covariance matrix to provide reliable forecasts for portfolio construction
Technical Appendix
Sample Correlation Matrix

- Sample period contains 713 weeks (01-03-01 to 8-27-14)
- Model contains $K=319$ local factors (for nine equity blocks)
- Compute sample covariance matrix ($F_0$) over $T=200$ weeks

$$F_{jk} = \frac{1}{T-1} \sum_t (f_{jt} - \bar{f}_j)(f_{kt} - \bar{f}_k)$$

- Let $S_0$ be a diagonal matrix of factor volatilities from $F_0$

$$C_0 = S_0^{-1}F_0S_0^{-1}$$

- $C_0$ provides an unbiased estimate of pairwise correlation
- However, $C_0$ is rank deficient (119 zero eigenvalues)
- $C_0$ falsely implies the existence of “riskless portfolios”
- $C_0$ is not suitable for portfolio optimization purposes
PCA Correlation Matrices (Global and Local)

- Transform the sample correlation matrix to diagonal basis
  \[ D_0 = U' \mathbf{C}_0 U \]
  Columns of \( U \) are eigenvectors of \( \mathbf{C}_0 \)

- Keep only the first \( J \) components, where \( J<T \) and \( J<K \)
  \[ \tilde{\mathbf{C}} = \tilde{U} \tilde{D} \tilde{U}' \]
  \( \tilde{U} \) is a \( K \times J \) matrix

- Compute the “idiosyncratic” variance
  \[ \Delta_{kk} = 1 - \text{diag}_k \left( \tilde{\mathbf{C}} \right) \]

- Scale PCA correlation matrix with official factor volatilities
  \[ \mathbf{F}_p = \mathbf{S} \mathbf{C}_p \mathbf{S} \]

- Global PCA refers to PCA technique on all local factors (\( K=319 \))
- Local PCA refers to applying PCA on the diagonal blocks

\[ \mathbf{C}_p = \tilde{\mathbf{U}} \tilde{D} \tilde{U}' + \Delta \]

PCA matrix

PCA covariance matrix
Random Matrix Theory (Global and Local)

- Consider the diagonal matrix $\hat{D}$:
  - First $J$ elements are the largest eigenvalues of sample correlation matrix
  - Remaining $K-J$ elements are the average of remaining eigenvalues

- Rotate back to the original basis
  \[ \hat{C} = U\hat{D}U' \]  
  Note: diagonal elements not equal to 1

- Scale rows and columns to recover 1 along the diagonals
  \[ C_R = \hat{S}^{-1}\hat{C}\hat{S}^{-1}, \text{ where } \hat{S}_{kk} = \sqrt{\hat{C}_{kk}} \]

- Scale RMT correlation matrix with official factor volatilities
  \[ F_R = SC_R S \]  
  RMT covariance matrix

- Global RMT refers to RMT technique on all local factors ($K=319$)
- Local RMT refers to applying RMT on the diagonal blocks
Time-Series Methods (Full Factor Set)

- Assume local factors are driven by a small set of global factors
  \[ f = gB + e \]
  - \( f \) is \( TxK \), \( g \) is \( TxJ \), \( B \) is \( JxK \), \( e \) is \( TxK \)
  - local factors, global factors, factor loadings, purely local

- For equities, the full set of explanatory variables is given by the factor returns of a global equity multi-factor model

- Factor loadings are estimated by time-series regression
  \[ B = (g'g)^{-1} g'f \]

- Define factor covariance matrices
  \[ G = \frac{g'g}{T - 1} \quad E = \frac{e'e}{T - 1} \quad D = \text{diag}(E) \]

- Local factor correlation matrix
  \[ F_{FS} = B'GB + D \quad \rightarrow \quad C_{FS} = S_{FS}^{-1}F_{FS}S_{FS}^{-1} \]

Time Series (Full Set)
Time-Series Methods (Partial Factor Set)

- Partial-set method mirrors full-set method, except each local factor is regressed on a small subset of global factors.
- For instance, the Japan Automobile factor might only be regressed on two global factors: Japan and Automobiles.
- This results in a sparse factor loadings matrix, \( \tilde{\mathbf{B}} \).
- Local factor covariance matrix is given by
  \[
  \mathbf{F}_{PS} = \tilde{\mathbf{B}}' \mathbf{G} \tilde{\mathbf{B}} + \mathbf{D}
  \]
- The correlation matrix is given by
  \[
  \mathbf{C}_{PS} = \mathbf{S}_{PS}^{-1} \mathbf{F}_{PS} \mathbf{S}_{PS}^{-1}
  \]
- Selection of relevant global factors:
  - May contain a significant subjective element.
  - Omission of important factors may lead to misestimation of risk.
Eigen-Adjusted Correlation Matrices (Local)

- Menchero, Wang, and Orr (2012) showed that eigenvalues of sample covariance matrix are systematically biased
  \[ D_0 = U'C_0U \]
  Columns of \( U \) are eigenvectors of \( C_0 \)

- Let \( \tilde{D}_0 \) denote the diagonal matrix of de-biased eigenvalues
- Perform reverse rotation to original basis:
  \[ \tilde{C} = U\tilde{D}_0U' \]
  Note: diagonal elements not equal to 1

- Scale rows and columns to recover 1 along the diagonals
  \[ C_E = \tilde{S}^{-1}\tilde{C}\tilde{S}^{-1} \]
  Eigen-adjusted correlation matrix

- Eigen-adjusted method is only applicable for the local blocks

Blended Correlation Matrices (Global and Local)

- Ledoit and Wolf (2003) showed that blending the sample covariance matrix with a factor model yielded optimized portfolios with lower out-of-sample volatility

- We blend the sample correlation matrix (using weight \( w \)) with the PCA correlation matrix (using \( J \) factors)

- Specify number of PCA factors by parameter \( \mu \), where \( J = \mu K \)

- Two-parameter model for correlation matrix:

\[
C_B(\mu, w) = wC_0 + (1-w)C_P(\mu)
\]

- Blending can be applied at either global or local level

Adjusted Correlation Matrices

- Local portfolio managers (e.g., US equities) want the “best” correlation matrix, known as the \textit{target} correlation matrix.

- Factor correlation matrix may be adjusted so that diagonal blocks agree with the target correlation matrix.

\[
C_T = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad \text{Target correlation matrix}
\]

\[
\hat{C} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \quad \text{Estimated correlation matrix}
\]

- Define adjustment matrix \( A \)

\[
A = \begin{bmatrix} C_{11}^{1/2} \hat{C}_{11}^{-1/2} & 0 \\ 0 & C_{22}^{1/2} \hat{C}_{22}^{-1/2} \end{bmatrix}
\]

\[
\hat{C}_A = A \hat{C} A' 
\]

- Diagonal blocks now agree with \( C_T \),

- Off-diagonal blocks given by: \( \hat{C}_A (2, 1) = C_{22}^{1/2} \hat{C}_{22}^{-1/2} \hat{C}_{21} \hat{C}_{11}^{1/2} \hat{C}_{12} \hat{C}_{11}^{-1/2} \).